The eleven dimensional supergravity equations, resolutions and Lefschetz fiber metrics

by

Xuwen Zhu

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Author . . . .
Department of Mathematics
Apr 5, 2015

Certified by . . . .
Richard B. Melrose
Simons Professor of Mathematics
Thesis Supervisor

Accepted by . . . .
William Minicozzi
Chairman, Department Committee on Graduate Studies

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Abstract

This thesis consists of three parts. In the first part, we study the eleven dimensional supergravity equations on $\mathbb{B}^7 \times \mathbb{S}^4$ considered as an edge manifold. We compute the indicial roots of the linearized system using the Hodge decomposition, and using the edge calculus and scattering theory we prove that the moduli space of solutions, near the Freund–Rubin states, is parameterized by three pairs of data on the bounding 6-sphere.

In the second part, we consider the family of constant curvature fiber metrics for a Lefschetz fibration with regular fibers of genus greater than one. A result of Obitsu and Wolpert is refined by showing that on an appropriate resolution of the total space, constructed by iterated blow-up, this family is log-smooth, i.e. polyhomogeneous with integral powers but possible multiplicities, at the preimage of the singular fibers in terms of parameters of size comparable to the length of the shrinking geodesic. This is joint work with Richard Melrose.

In the third part, the resolution of a compact group action in the sense described by Albin and Melrose is applied to the conjugation action by the unitary group on self-adjoint matrices. It is shown that the eigenvalues are smooth on the resolved space and that the trivial tautological bundle smoothly decomposes into the direct sum of global one-dimensional eigenspaces.

Thesis Supervisor: Richard B. Melrose
Title: Simons Professor of Mathematics

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First and foremost, I would like to express my sincere gratitude to my advisor, Richard Melrose, for introducing me into this field, suggesting interesting problems and providing tremendous support for my Ph.D study and research. His deep insight, patient guidance and encouragement over the years has been invaluable to me. As noted below, this thesis is in part joint work with him.

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Chapter 1

Introduction

Microlocal analysis has a lot of applications in partial differential equations and analysis of problems with geometry and physics background. In this thesis, we apply microlocal techniques and the theory of pseudo-differential operators to two problems on noncompact and singular manifolds. The first one is the eleven dimensional supergravity equations on edge manifolds for which I give a characterization of all the solutions near the Freund–Rubin solution [47]. The second project, in collaboration with Richard Melrose, we work on the complete expansion of the constant scalar curvature fiber metric in the case of a Lefschetz fibration [31], which arises naturally as the singular behavior across the divisors introduced in the Deligne–Mumford compactification of the moduli space of Riemann surfaces.

To get information of operators in a singular geometry setting, people study the Schwarz kernel and the behavior of the model operators on the double space. To study the behavior of the kernel and construct parametrices on this double space, blow up action is introduced [34, 32] and this approach has been utilized to solve many geometric problems [35, 26, 27, 24, 44]. The third part of this thesis contains an example of resolutions, which is an application of the resolution of a compact group action on a compact manifold described by Albin and Melrose [1], to the case of $U(n)$ action on self-adjoint matrices [48].

1.1 Eleven dimensional supergravity theory on edge manifolds

Supergravity theories arise as the representations of super Lie algebras in various dimensions. They can be viewed as low energy approximations to string theory with classical equations of motion. In particular, the case of dimension eleven has been studied by physicists since the 1970s [43, 42, 6, 2]. It was shown that there is a unique system in eleven dimensions and the theories in lower dimensions can be obtained from it through dimension reduction [37]. Since it is related to the AdS/CFT correspondence and brane dynamics, this subject has recently attracted more attention [3, 45].

We are interested in a particular case, namely, the bosonic sector of eleven dimensional supergravity theory. The nonlinear system couples a metric g (gravity) and a 4-form F (extra field), and is derived from a Lagrangian constructed on the eleven dimensional manifold $X = \mathbb{B}^7 \times \mathbb{S}^4$:

$$L(g, F) = \int_X R dV_g - \frac{1}{2}\left(\int_X F \wedge *F + \int_X \frac{1}{3} A \wedge F \wedge F\right) \tag{1.1}$$

where A satisfies $dA = F$. The first term is the classical Einstein–Hilbert action term, while the second and the third terms are, respectively, of Yang–Mills and Chern-Simons type for a field. The supergravity system, the variational equation for (1.1) is

$$\begin{aligned} &R_{\alpha\beta} = \tfrac{1}{12}\left(F_{\alpha\gamma_1\gamma_2\gamma_3} F_\beta^{\gamma_1\gamma_2\gamma_3} - \tfrac{1}{12} F_{\gamma_1\gamma_2\gamma_3\gamma_4} F^{\gamma_1\gamma_2\gamma_3\gamma_4} g_{\alpha\beta}\right) \\ &d * F = -\tfrac{1}{2} F \wedge F \\ &dF = 0. \end{aligned} \tag{1.2}$$

As a special case, there is a family of solutions to the full system given by the product of a scaled spherical metric on $\mathbb{S}^4$ and an Einstein metric h on $\mathbb{B}^7$ with Ricci curvature satisfying $\mathrm{Ric}(h) = -c^2 h$. The product solutions are given by

$$g = h \times \frac{9}{c^2} g_{\mathbb{S}^4}, \quad F = c dV_{\mathbb{S}^4}, \quad \forall c \in \mathbb{R}. \tag{1.3}$$

In particular for $c = 6$ these are known as Freund–Rubin solutions [11] with 1/4 of the standard $\mathbb{S}^4$ metric and an Einstein metric on $\mathbb{H}^7$. When we restrict the search of solutions to a product metric, the following theorem by Graham–Lee [15] showed that the existence of Poincaré–Einstein solutions near the hyperbolic metric prescribed by data at conformal infinity:

Theorem 1.1 ([6])**.** *Let $M = \mathbb{B}^{n+1}$ be the unit ball and $\hat{h}$ the standard metric on $\mathbb{S}^n$. For any smooth Riemannian metric $\hat{g}$ on $\mathbb{S}^n$ which is sufficiently close to $\hat{h}$ in $C^{2,\alpha}$ norm if $n > 4$, or $C^{3,\alpha}$ norm if $n = 3$, for some $0 < \alpha < 1$, there exists a smooth metric g on the interior of M, with a C^0 conformal compactification with conformal infinity $[\hat{g}]$ and*

$$\mathrm{Ric}(g) = -ng.$$

Combined with (1.3), the 4-form F being a multiple of the 4-sphere volume form gives a family of solutions parametrized by conformal metrics on the bounding 6-sphere. This product solution corresponds to the edge structure in the sense of Mazzeo [25, 23]. An edge structure is defined on a manifold M where the boundary has a fibration over a compact manifold as follows,

$$\begin{array}{ccc} \pi : F & \longrightarrow & \partial M \\ & & \downarrow \\ & & B \end{array} \tag{1.4}$$

which, is our case, is the product fibration

$$\begin{array}{ccc} \pi : \mathbb{S}^4 & \longrightarrow & \partial M \\ & & \downarrow \\ & & \mathbb{S}^6 \end{array}$$

The space of edge vector fields $\mathcal{V}_e(M)$ is a Lie algebra consisting of those smooth vector fields on M which are tangent to the boundary and such that the induced vector fields on the boundary are tangent to the fibres of π. Let $(x, y_1, y_2, ...y_6)$ be coordinates of the upper half space model for hyperbolic space $\mathbb{H}^7$, and z_j be coordinates on the sphere $\mathbb{S}^4$. Then locally $\mathcal{V}_e$ is spanned by

$$x\partial_x, x\partial_y, \partial_z.$$

The edge forms are the dual to the edge vector fields $\mathcal{V}_e$, with a basis:

$$\frac{dx}{x}, \frac{dy}{x}, dz.$$

The edge tensors and co-tensors are the products of those basis forms, and the solutions we look for are in the sections of the edge bundles.

In Mazzeo's paper [25] the Fredholm property of certain elliptic edge operators has been discussed. It is related to the invertibility of the corresponding normal operator $N(L)$, which is the lift of the operator to the front face of the double stretched space X_e^2. If we write the edge operator in local coordinates as:

$$L = \sum_{j+|\alpha|+|\beta|\leq m} a_{j,\alpha,\beta}(x,y,z)(x\partial_x)^j(x\partial_y)^\alpha(\partial_z)^\beta, \tag{1.5}$$

then the normal operator is

$$N(L) = \sum_{j+|\alpha|+|\beta|\leq m} a(0,\tilde{y},z)(s\partial_s)^j(s\partial_u)^\alpha\partial_z^\beta, \tag{1.6}$$

where $(s, u, \tilde{y}, z, \tilde{z})$ is a coordinate on the front face of X_e^2. The invertibility of the normal operator is in turn related to its action on functions polyhomogeneous at the left boundary of X_e^2, of which the expansion is determined by the indicial operator, which by definition is

$$I_p[P](s)v = x^{-s}P(x^s v)|_{\pi^{-1}(p)}, \tag{1.7}$$

where p is a point in the base and s is a complex number. and in local coordinate $I(L)$ is written out (using a conjugation of Mellin transform M_{is}) as

$$I[L](s) = M_{is}\left(\sum_{j+|\beta|\leq m} a_{j,0,\beta}(0,y,z)(s\partial_s)^j\partial_z^\beta\right)M_{is}^{-1} \tag{1.8}$$

The inverse of the indicial operator $I(L)(\theta)^{-1}$ exists and is meromorphic on the complement of a discrete set $\operatorname{spec}_b L$, which is the indicial roots of L. Those indicia roots provide information of the operator, and more precisely, the parametrix of an edge operator is constructed on the stretched double product where the Schwartz kernel is lifted with polyhomogeneous expansions.

Theorem 1.2 ([25]). *If an elliptic edge operator $L \in \operatorname{Diff}_e^m(M)$ has constant indicial roots over the boundary and its normal operator L_0 and its adjoint L_0^t has the unique continuation property, then L is essentially injective (resp. surjective) for a weight parameter $\delta \notin \Lambda = \{\operatorname{Re}\theta + 1/2 : \theta \in \operatorname{spec}_b L\}$ and $\delta \gg 0$ (resp. $\delta \ll 0$), and in either case has closed range.*

With the edge vector fields one can define the edge Sobolev spaces

$$H_e^s(M) = \{u \in L^2(M) \mid \mathcal{V}_e^k u \in L^2(M), 0 \leq k \leq s\}. \tag{1.9}$$

For purpose of regularity we are also interested in hybrid spaces with additional tangential regularity. The existence of solutions with infinite smooth b-regularity gives the solution with polyhomogeneous expansions. Therefore we set the Sobolev space with boundary and edge regularity as:

$$H_{e,b}^{s,k}(M) = \{u \in H_e^s(M) \mid \mathcal{V}_b^i u \in H_e^s(M), 0 \leq i \leq k\}$$

By the commuting relation $[\mathcal{V}_b, \mathcal{V}_e] \subset \mathcal{V}_b$, $H_{e,b}^{s,k}(M)$ is well defined, that is, independent to the order of applying edge and b-vector fields. These Sobolev spaces are defined so that edge operators maps between suitable spaces, i.e., for any m-th order edge operator $P \in \operatorname{Diff}_e^m M$,

$$P : H_{e,b}^{s,k}(M) \to H_{e,b}^{s-m,k}(M), m \leq s. \tag{1.10}$$

Following the idea of Graham–Lee [15] of constructing solutions that are close to the hyperbolic metric, we are interested in those solutions to (1.2) that are quasi-isometric to the Freund–Rubin solution in the edge class, that is, as sections of edge bundles $\mathrm{Sym}^2({}^eTM)\oplus{}^e\bigwedge^4 T^*M$. Kantor in his thesis [20] first considered this problem and constructed a family of solutions to the linearized equations, which correspond to change of the 4-form along one particular direction. Our result is a generalization of the results of Graham–Lee and Kantor, in that we considered the variation of the metric and the 4-form together.

The structure of the proof is as follows. In section 2.2 we fix the gauge of this system, using the DeTurck gauge-breaking term $\phi(t,g)$ introduced in [15] and show that by adding this gauge term we get an operator Q, which is a map on the space of symmetric 2-tensors and closed 4-forms:

$$Q : S^2(T^*M)\oplus\textstyle\bigwedge^4_{cl}(T^*M)\to S^2(T^*M)\oplus\bigwedge^4_{cl}(T^*M)$$

$$\begin{pmatrix} g \\ F \end{pmatrix}\mapsto\begin{pmatrix} \mathrm{Ric}(g)-\phi(t,g)-F\circ F \\ d*(d*F+\frac{1}{2}F\wedge F). \end{pmatrix} \tag{1.11}$$

The solution to the gauged equation uniquely determines a solution to the original equations.

In section 2.3 we compute the indicial roots of the linearized gauged equations. The system splits according to the degree of forms on the product manifold, and further breaks down into blocks under the Hodge decomposition on the 4-sphere. The indicial roots apppear in pairs, symmetric around the line $\mathrm{Re}\, z = 3$, and are parametrized by the eigenvalues of the 4-sphere (see Figure 2-1 for the indicial roots distribution). Then according to different behaviors of the indicial roots we use different strategies. For large eigenvalues, for which the indicial roots separate away from the L^2 line, a parametrix is constucted using the small edge calculus introduced by Mazzeo [25], showing that the operator is an isomorpism. The Fredholm property of the smaller eigenvalues that are still separated away from L^2 line is individually discussed in a similar manner using normal operator construction; then we use the fact there are no finite-dimensional $SO(7)$ invariant sections in L^2 on hyperbolic space for symmetric tensors and forms [7], to show that the operator is injective on any space that is contained in L^2. For the three pairs with real part of indicial roots equal to 3, the projection of the operator becomes a 0-problem and we follow the method in Mazzeo–Melrose [29] and Guillarmou [18] to construct two generalized inverses $R^\pm = \lim_{\epsilon\to 0}(dQ \pm i\epsilon)^{-1}$. Then we use scattering matrix construction on hyperbolic space given by Graham–Zworski [16] and Guillarmou–Naud [19] to show explicitly that real-valued kernel of the linearized operator is prescribed by three pairs of data on the boundary 6-sphere.

In section 2.4 we apply the implicit function theorem, using the fact that the nonlinear terms are all quadratic, so the nonlinear solutions are also parametrized by these three terms.

Now we state the main theorem. Let

$$V_1 := \{v_1 \in C^\infty(\mathbb{S}^6; \textstyle\bigwedge^3 T^*\mathbb{S}^6) : *_{\mathbb{S}^6} v_1 = iv_1\}.$$

Let $V_2^\pm$, $V_3^\pm$ be the smooth functions on the 6-sphere tensored with a finite dimensional 1-form space on $\mathbb{S}^4$:

$$V_2 := \{v_2 \otimes \xi_{16} : v_2 \in C^\infty(\mathbb{S}^6; \mathbb{R}), \xi_{16} \in E_{16}^{cl}(\mathbb{S}^4)\}$$

$$V_3 := \{v_3 \otimes \xi_{40} : v_3 \in C^\infty(\mathbb{S}^6; \mathbb{R}), \xi_{40} \in E_{40}^{cl}(\mathbb{S}^4)\}$$

where $E_{16}^{cl}(\mathbb{S}^4)$ and E_{40}^{cl} are closed 1-forms with eigenvalue 16 and 40 on the 4-sphere.

We also require three numbers that define the leading term in the expansion of the solution, which come from indicial roots:

$$\theta_1^\pm = 3 \pm 6i, \theta_2^\pm = 3 \pm i\sqrt{21116145}/1655, \theta_3 = 3 \pm i3\sqrt{582842}/20098. \qquad (1.12)$$

If we fix an element $[\hat{h}]$ in the conformal boundary data to the leading order, the solution is parametrized by a small perturbation from the data on the bundle $C^\infty(\mathbb{S}^6; \oplus_{i=1}^3 V_i)$. The metric part of the solution to the leading order is given by the conformal infinity $[\hat{h}]$, whereas the form part to the leading order is given by the oscillatory data $v_i^+ x^{\theta_i^+} + S_i(v_i)x^{\theta^-}$.

To state the theorem, we also give the following notations. Denote

$$u_0 := (g_{\mathbb{H}^7} \times \frac{1}{4} g_{\mathbb{S}^4}, 6\,\mathrm{Vol}_{\mathbb{S}^4})$$

and $[\hat{h}]$ is close to $\hat{g}_0$ as metric on $\mathbb{S}^6$. A small neighborhood in the bundle is given by

$$U \subset C^\infty(\mathbb{S}^6; \oplus_{i=1}^3 V_i)$$

Theorem 1.3. *For $\delta \in (0,1)$, $s \geq 2$ and $k \gg 0$, in the space of solutions to $Q(u) = 0$ in $x^{3-\delta} H_b^\infty(M; W)$, a neighborhood of u_0 is smoothly parametrized by $[\hat{h}]$ and U. For a smooth section $v \in U$ with a sufficiently small H^k norm and a $[\hat{h}]$, there is a unique $g \in x^{-\delta} H_b^\infty(M; \mathrm{Sym}^2({}^e T^* M))$ and a 4-form $F \in x^{-\delta} H_b^\infty(M; {}^e \bigwedge^4 (T^* M))$ prescribed by those data, such that $(g - h, F - V_0) \in x^{-\delta} H_{e,b}^{s;k}(M; W)$ and $Q(u) = 0$.*

1.2 Resolution of the canonical fiber metrics for a Lefschetz fibration

In this project joint with Richard Melrose, we give a complete description of the behavior of the constant scalar curvature fiber metric on a Lefschetz fibration with fiber genus ≥ 2. For a Riemann surface with $g \geq 2$, the classical uniformization theorem guarantees the existence of a metric with constant scalar curvature -1. One may ask the question about the existence and behavior of a constant scalar curvature metric if the geometry becomes singular, namely, we take a nontrivial geodesic cycle and let its length go to zero, which is illustrated in Figure 1-1.

This fits naturally with the setting of a Lefschetz fibration. The class of Lefschetz fibrations we consider is for a compact connected almost-complex 4-manifold W and a smooth map, with complex fibers F, to a Riemann surface Z

$$\begin{array}{ccc} \psi : F & \longrightarrow & W \\ & & \downarrow \\ & & Z \end{array} \tag{1.13}$$

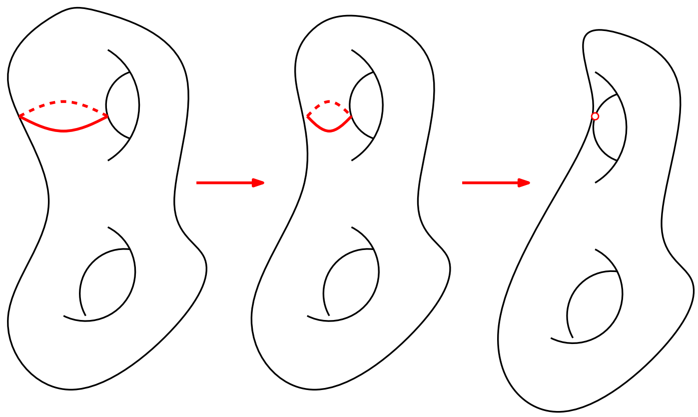

Figure 1-1: Degenerating surfaces with a geodesic cycle shrinking to a point

which is pseudo-holomorphic, has surjective differential outside a finite subset of W and near each of these singular points is reducible to the following normal crossing model:

$$P_t = \left\{(z, w, t) \in \mathbb{C}^3; zw = t,\ |z| \leq 1,\ |w| \leq 1,\ |t| \leq \frac{1}{2}\right\} \ni (z, w, t) \longrightarrow \mathbb{D}_{\frac{1}{2}} = \left\{t \in \mathbb{C}; |t| \leq \frac{1}{2}\right\}. \quad (1.14)$$

Lefschetz fibrations play an important role in 4-manifold theory. Donaldson [8] showed that a 4-dimensional simply-connected compact symplectic manifold admits a Lefschetz fibration over a sphere up to a stabilization, and Gompf showed the converse [14].

Obitsu and Wolpert [39] [46] studied a degenerating family of Riemann surfaces R_t where near each singularity the model is P_t and the metric is given by the plumbing metric

$$\begin{aligned} ds^2_{P_t} &= \left(\frac{\pi \log |z|}{\log |t|} \csc \frac{\pi \log |z|}{\log |t|}\right)^2 ds^2_0, \\ ds^2_0 &= \left(\frac{|dz|}{|z| \log |z|}\right)^2. \end{aligned} \quad (1.15)$$

Fiberwise it has constant scalar curvature -1, and approaches the singular metric ds^2_0 on the cylinder as t tends to 0. By grafting with the regular parts, they constructed the expansion of metrics on the global manifold.

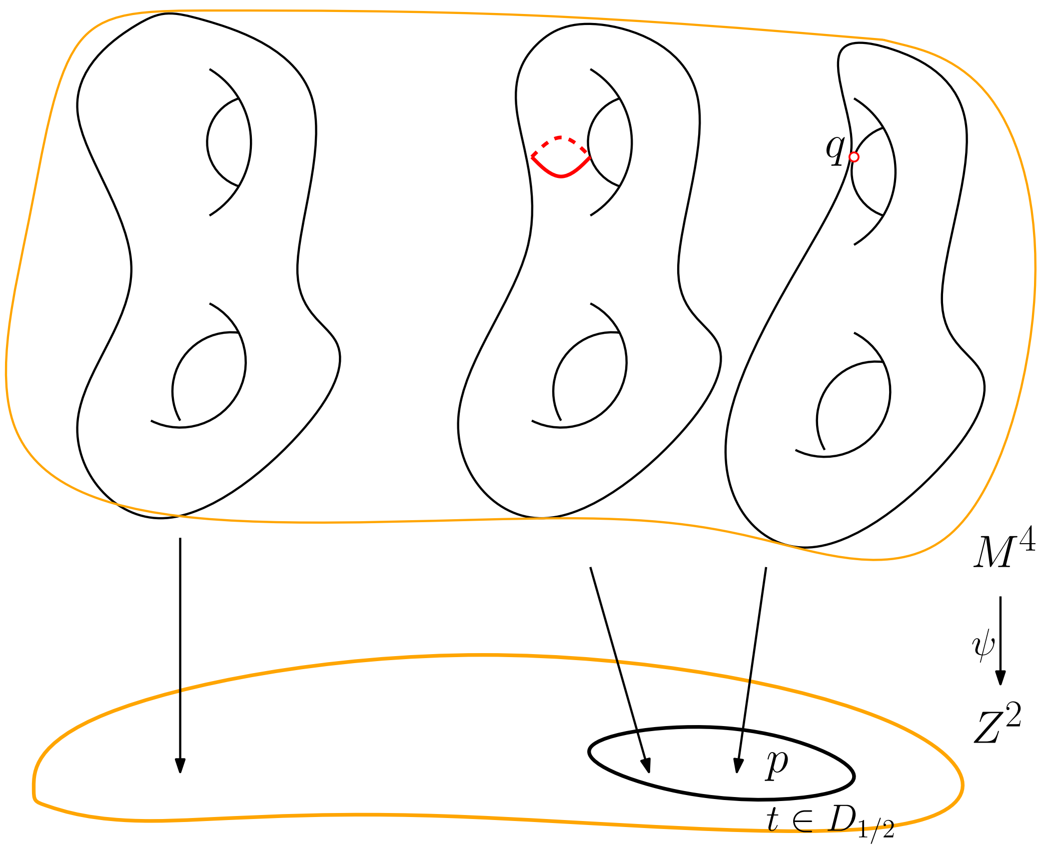

Figure 1-2: Lefschetz fibration $\psi : M \to Z$

Theorem 1.4 ([39], [46]). *Let ds^2_{cc} be the hyperbolic metric on the degenerated family R_t with m vanishing cycles, Δ the associated Laplacian, and ds_{pl} the plumbing metric that comes from gluing $ds^2_{P_t}$ with the regular part, then the metric has the following expansion*

$$ds^2_{cc} = ds^2_{pl}\left(1 - \frac{\pi^2}{3}\sum_{j=1}^{m}\left(\frac{1}{\log|t_j|}\right)^2(\Delta-2)^{-1}(\Lambda(z_j)+\Lambda(w_j)) + O\left(\sum\left(\frac{1}{\log|t_j|}\right)^4\right)\right) \tag{1.16}$$

where the function Λ is given by $\Lambda(z_j) = (s_z^4\chi_{\pi^{-1}\mathbb{D}_{1/2}})_{s_z}$, $s_z = \log|z_j|$.

Our result is a generalization of Theorem 1.4 by showing that, in the resolved plumbing space, g_{cc} is conformal to g_{pl}, where the conformal factor has a complete expansion in the variable $\log|t|$.

In order for the metric to be smooth, we introduce the resolved plumbing space which involves three steps of construction. The first step in the resolution is the blow up, in the real sense, of the singular fibers; this is well-defined in view of the transversality of the self-instersection but results in a tied manifold since the boundary faces are not globally embedded. The second step is to replace the $\mathcal{C}^\infty$ structure by its logarithmic weakening, i.e. replacing each (local) boundary defining function x by

$$\operatorname{ilog} x = (\log x^{-1})^{-1}.$$

This gives a new tied manifold mapping smoothly to the previous one by a homeomorphism. These two steps can be thought of in combination as the 'logarithmic blow up' of the singular fibers. The final step is to blow up the corners, of codimension two, in the preimages of the singular fibers. This results in a manifold with corners, M_{mr}, with the two boundary hypersurfaces denoted B_{I}, resolving the singular fiber, and B_{II} arising at the final stage of the resolution. The parameter space Z is similarly resolved to a manifold with corners by the logarithmic blow up of each of the singular points.

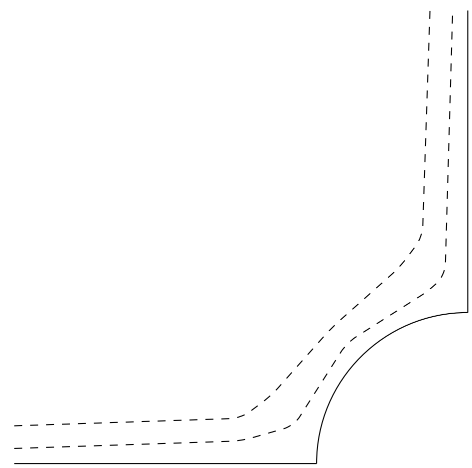

Figure 1-3: The metric resolution

It is shown below that the Lefschetz fibration lifts to a smooth map

$$M_{\mathrm{mr}} \xrightarrow{\psi_{\mathrm{mr}}} Z_{\mathrm{mr}} \tag{1.17}$$

which is a b-fibration. In particular it follows from this that smooth vector fields on M_{mr} which are tangent to all boundaries and to the fibers of ψ_{mr} form the sections of a smooth vector subbundle of ${}^{\mathrm{b}}TM_{\mathrm{mr}}$ of rank two. The boundary hypersurface B_{II} has a preferred class of boundary defining functions, an element of which is denoted ρ_{II}, arising from the logarithmic nature of the resolution, and this allows a Lie algebra of vector fields to be defined by

$$V \in \mathcal{C}^\infty(M_{\mathrm{mr}}; {}^{\mathrm{b}}TM_{\mathrm{mr}}),\ V\psi^*\mathcal{C}^\infty(Z_{\mathrm{mr}}) = 0,\ V\rho_{\mathrm{II}} \in \rho_{\mathrm{II}}^2\mathcal{C}^\infty(M_{\mathrm{mr}}). \tag{1.18}$$

The possibly singular vector fields of the form $\rho_{\mathrm{II}}^{-1}V$, with V as in (3.4), also form all the sections of a smooth vector bundle, denoted ${}^{L}TM_{\mathrm{mr}}$. This vector bundle inherits a complex structure and hence has a smooth Hermitian metric, which is unique up to a positive smooth conformal factor on M_{mr}. The main result of this paper is:

Theorem 1.5. *The fiber metrics of fixed constant curvature on a Lefschetz fibration, in the sense discussed above, extend to a continuous Hermitian metric on* ${}^{L}TM_{\mathrm{mr}}$ *which is related to a smooth Hermitian metric on this complex line bundle by a log-*

smooth conformal factor.

The plumbing metric can be extended ('grafted' as in [39]) to give an Hermitian metric on ${}^{L}TM_{\mathrm{mr}}$ which has curvature R equal to -1 near B_{II} and to second order at B_{I}. We prove the theorem above by constructing the conformal factor e^{2f} for this metric which satisfies the curvature equation, ensuring that the new metric has curvature -1 :

$$(\Delta + 2)f + (R + 1) = -e^{2f} + 1 + 2f = O(f^2). \tag{1.19}$$

This equation is first solved in the sense of formal power series (with logarithms) at both boundaries, B_{I} and B_{II}, which gives us an approximate solution f_0 with

$$-\Delta f_0 = R + e^{2f_0} + g,\ \ g \in s_t^{\infty}\mathcal{C}^{\infty}(M_{\mathrm{mr}}).$$

Then a solution $f = f_0 + \tilde{f}$ to (3.9) amounts to solving

$$\tilde{f} = -(\Delta + 2)^{-1}\left(2\tilde{f}(e^{2f_0} - 1) + e^{2f_0}(e^{2\tilde{f}} - 1 - 2\tilde{f}) - g\right) = K(\tilde{f}).$$

Here the non-linear operator K is at least quadratic in $\tilde{f}$ and the boundedness of $(\Delta + 2)^{-1}$ on $\rho_{\mathrm{II}}^{-\frac{1}{2}} H_{\mathrm{b}}^{M}(M_{\mathrm{mr}})$ for all M allow the Inverse Function Theorem to be applied to show that $\tilde{f} \in s_t^{\infty}\mathcal{C}^{\infty}(M_{\mathrm{mr}})$ and hence that f itself is log-smooth.

In §3.1 the model space and metric are analysed and in §3.2 the global resolution is described and the proof of the theorem above is outlined. The linearized model involves the inverse of $\Delta+2$ for the Laplacian on the fibers and the uniform behavior, at the singular fibers, of this operator is explained in §3.3. The solution of the curvature problem in formal power series is discussed in §3.4 and using this the regularity of the fiber metric is shown in §3.5.

1.3 Resolution of eigenvalues

This project is an explicit construction for an example of the Albin–Melrose resolution of compact group action on a compact manifold [1]. Such resolution of group actions is interesting because on the resolved space, the group-equivariant objects are well-defined and smooth. Albin–Melrose gave a general scheme of how to resolve the group actions according to the index of isotropy types, namely, an iterative scheme where the smallest isotropy type is blown up and then the next level of stratum could be uniform.

Theorem 1.6 ([1]). *A compact manifold with corners, M, with a smooth boundary intersection free action by a compact Lie group G, has a canonical full resolution, $Y(M)$, obtained by iterative blow-up of minimal isotropy types.*

Here we consider the action of the unitary group $U(n)$ on the space of n-dimensional self-adjoint matrices $S(n)$ and construct the resolved space $\widehat{S(n)}$ with a fixed isotropy type, that is, $\widehat{S(n)}/U(n)$ is a smooth manifold. We introduce the following two definitions of resolutions:

Definition 1.1 (eigenresolution). *By an eigenresolution of S, we mean a manifold with corners $\hat{S}$, with a surjective smooth map $\beta : \hat{S} \to S$ such that the self-adjoint matrices have a smooth (local) diagonalization when lifted to $\hat{S}$, with eigenvalues lifted to smooth functions on $\hat{S}$.*

Definition 1.2 (full eigenresolution). *A full eigenresolution is an eigenresolution with global eigenbundles. The eigenvalues are lifted to n smooth functions f_i on $\hat{S}$, and the trivial n-dimensional complex vector bundle on $\hat{S}$ is decomposed into n smooth line bundles $\hat{S} \times \mathbb{C}^n = \bigoplus_{i=1}^n E_i$ such that $\beta(x)v_i = f_i(x)v_i, \forall\ v_i \in E_i(x), \forall\ x \in \hat{S}$.*

The matrices belong to different isotropy types, in this case, are indexed by the clustering of eigenvalues. First the two dimensional matrix case is explicitly computed, where the only singularities are the multiples of the identity matrix. Then for higher dimensional matrices, a local product structure using the Grassmannian is described, so we show that when there is a uniform spectral gap in the neighborhood, there is a local product decomposition into two lower rank matrices and a neighborhood in the Grassmannian. Then using this decomposition, we show that the blow up action can be done iteratively, each time blowing up the smallest isotropy type, which in our case is the matrices with the smallest number of different eigenvalues.

Theorem 1.7 ([48]). *The iterative blow up of the isotropy types in S (in an order compatible with inclusion of the conjugation class of the isotropy group) yields an eigenresolution. In particular, radial blow up gives a full eigenresolution.*

We also discuss the difference between radial blow up and projective ones. We show in the example of two dimensional matrices that only after radial blow up the trivial $\mathbb{C}^2$ bundle splits into two global line bundles, while in the projective case there is no global splitting, which gives an example of the discussions in [1] that projective blow up does not induce a global resolution.

Chapter 2

The eleven dimensional supergravity equations on edge manifolds

2.1 Introduction

Supergravity is a theory of local supersymmetry, which arises in the representations of super Lie algebras. Nahm [37] showed that the dimension of the system is at most eleven in order for the system to be physical, and in this dimension if the system exists then it would be unique. The existence of such systems was shown later by Cremmer–Scherk [6] by constructing a specific system. Recently, Witten [45] showed that under AdS/CFT correspondence the M-theory is related to the 11-dimensional supergravity system, and as a result people start to work on this subject again [3]. Systems of lower dimensions can be obtained by dimensional reduction, which breaks into many smaller subfields, and in general there are many such systems. The full eleven dimensional case, with only two fields, is in many ways the simplest to consider.

A supergravity system is a low energy approximation to string theories, and can be viewed as a generalization of Einstein's equation:$R_{\alpha\beta} = ng_{\alpha\beta}$. We are specifically interested in the bosonic sections in the supergravity theory, which is a system of equations on the 11-dimensional product manifold $M = \mathbb{B}^7 \times \mathbb{S}^4$ that solves for a metric, g, and a 4-form, F. Derived as the variational equations from a Lagrangian, the supergravity equations are as follows:

$$\begin{aligned} &R_{\alpha\beta} = \tfrac{1}{12}(F_{\alpha\gamma_1\gamma_2\gamma_3}F_{\beta}^{\gamma_1\gamma_2\gamma_3} - \tfrac{1}{12}F_{\gamma_1\gamma_2\gamma_3\gamma_4}F^{\gamma_1\gamma_2\gamma_3\gamma_4}g_{\alpha\beta}) \\ &d * F = -\tfrac{1}{2}F \wedge F \\ &dF = 0 \end{aligned} \tag{2.1}$$

The nonlinear supergravity operator has an edge structure in the sense of Mazzeo [25], which is a natural generalization in the context of the product of a conformally compact manifold and a compact manifold. We consider those solutions that are sections of the edge bundles, which are rescalings of the usual form bundles. The Fredholm property of certain elliptic edge operators is related to the invertibility of the corresponding normal operator $N(L)$, which is the lift of the operator to the front face of the double stretched space X_e^2. The invertibility of the normal operator is in turn related to its action on functions polyhomogeneous at the left boundary of X_e^2, which is determined by the indicial operator. The inverse of the indicial operator $I_\theta(L)^{-1}$ exists and is meromorphic on the complement of a discrete set $\operatorname{spec}_b L$, which is the indicial roots of L. In this way the indicial operator as a model on the boundary determines the leading order expansion of the solution.

One solution for this system is given by a product of the round sphere with a Poincaré–Einstein metric on $\mathbb{B}^7$ with a volume form on the 4-sphere, in particular the Freund–Rubin solution [11] is contained in this class. Recall that a Poincaré–Einstein manifold is one that satisfies the vacuum Einstein equation and has a conformal boundary. In the paper by Graham and Lee [15], they constructed the solutions which are $C^{n-1,\gamma}$ close to the hyperbolic metric on the ball $\mathbb{B}^n$ near the boundary, and showed that every such perturbation is prescribed by the conformal data on the boundary sphere. We will follow a similar idea here for the equation (2.1), replacing the nonlinear Ricci curvature operator by the supergravity operator, considering its linearization around one of the product solutions, and using a perturbation argument to show that all the solutions nearby are determined by the metric and form data on the boundary.

Kantor studied this problem in his thesis [20], where he computed the indicial roots of the system and produced one family of solutions by varying along a specific direction of the form. Here we use a different decomposition that gives the same indicial roots and show that all the solutions nearby are prescribed by boundary data for the linearized operator, more specifically, the indicial kernels corresponding to three pairs of special indicial roots.

2.1.1 Equations derived from the Lagrangian

The 11-dimension supergravity theory contains the following information on an 11-dimensional manifold M: gravity metric $g \in \mathrm{Sym}^2(M)$ and a 4-form $F \in \bigwedge^4(M)$. In this theory, the Lagrangian L is defined as

$$L(g, A) = \int_M R dV_g - \frac{1}{2}\Big(\int_M F \wedge *F + \int_M \frac{1}{3} A \wedge F \wedge F \Big) \tag{2.2}$$

Here R is the scalar curvature of the metric g, A is a 3-form such that F is the field strength $F = dA$. The first term is the calssical Einstein-Hilbert action term, where the second and the third one are respectively Yang-Mills type and Maxwell type term for a field. Note here we are only interested in the equations derived from the variation of Lagrangian, therefore A needs not to be globally defined since we only need dA because the variation only depends on dA:

$$\delta_{A_i}\Big(\int_{U_i} A_i \wedge F \wedge F \Big) = 3 \int_{U_i} \delta A_i \wedge F \wedge F \tag{2.3}$$

which shows that the variation is $F \wedge F$ which does not depend on A_i.

The supergravity equations, derived from the Lagrangian above, are (2.1). To deal with the fact the Ricci operator is not elliptic, we follow [15] and add a gauge breaking term $\phi(g, t) = \delta_g^* g \Delta_{gt} Id$ to the first equation. Then we apply $d*$ to the 2nd equation, and combine this with the third equation to obtain the gauged supergravity system:

$$Q : S^2(T^*M) \oplus \bigwedge^4(M) \to S^2(T^*M) \oplus \bigwedge^4(M)$$

$$\begin{pmatrix} g \\ F \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{Ric}(g) - \phi_{(g,t)} - F \circ F \\ d * (d * F + \frac{1}{2} F \wedge F) \end{pmatrix} \tag{2.4}$$

which is the nonlinear system we will be studying.

2.1.2 Edge metric and edge Sobolev space

Edge differential and pseudodifferential operators were formally introduced by Mazzeo [25]. The general setting is a compact manifold with boundary, M, where the boundary has in addition a fibration

$$\pi : \partial M \to B,$$

with typical fiber F. In the setting considered here, $M = \mathbb{B}^7 \times \mathbb{S}^4$ is the product of a seven dimensional closed ball identified as hyperbolic space and a four-dimensional sphere. The fibration here identifies the four-sphere as fibre:

$$\pi : \partial(\mathbb{H}^7 \times \mathbb{S}^4) = \mathbb{S}^6 \times \mathbb{S}^4 \to \mathbb{S}^6.$$

The space of edge vector fields $\mathcal{V}_e(M)$ is a Lie algebra consisting of those smooth vector fields on M which are tangent to the boundary and such that the induced vector field on the boundary is tangent to the fibre of π. Another vector field Lie algebra we will be using is $\mathcal{V}_b$ which is the space of all smooth vector field tangent to the boundary. As a consequence,

$$\mathcal{V}_e \subset \mathcal{V}_b, \quad [\mathcal{V}_e, \mathcal{V}_b] \subset \mathcal{V}_b. \tag{2.5}$$

Let $(x, y_1, y_2, ...y_6)$ be coordinates of the upper half space model for hyperbolic space $\mathbb{H}^7$, and z_j be coordinates on the sphere $\mathbb{S}^4$. Then locally $\mathcal{V}_b$ is spanned by $x\partial_x, \partial_y, \partial_z$, while $\mathcal{V}_e$ is spanned by

$$x\partial_x, x\partial_y, \partial_z.$$

The edge forms are the dual to the edge vector fields $\mathcal{V}_e$, with a basis:

$$\Big(\frac{dx}{x}, \frac{dy}{x}, dz\Big).$$

The 2-tensor bundle is formed by the tensor product of the basis forms. Edge differential operators form the linear span of products of edge vector fields over smooth functions. Denote the set of m-th order edge operator as $\mathrm{Diff}_e^m(M)$. We will see that the supergravity operator Q is a nonlinear edge differental operator.

The edge-Sobolev spaces are given by

$$H_e^s(M) = \{u \in L^2(M) | V_e^k u \in L^2(M), 0 \le k \le s\}.$$

However, for purpose of regularity we are also interested in hybrid spaces with additional tangential regularity. The exsitence of solutions with infinite smooth b-regularity gives the solution with polyhomogeneous expansions. Therefore we set the Sobolev space with boundary and edge regularity as:

$$H_{e,b}^{s,k}(M) = \{u \in H_e^s(M) | V_b^i u \in H_e^s(M), 0 \le i \le k\}$$

By the commuting relation (2.5), $H_{e,b}^{s,k}(M)$ is well defined, that is, independent to the order of applying edge and b-vector fields.

These Sobolev spaces are defined so that edge operators maps between suitable spaces, i.e., for any m-th order edge operator $P \in \mathrm{Diff}_e^m M$,

$$P : H_{e,b}^{s,k}(M) \to H_{e,b}^{s-m,k}(M), m \le s. \tag{2.6}$$

for which the proof is contained in section 2.5.

2.1.3 Poincaré–Einstein metric on $\mathbb{B}^7$

Now let us go back to the Poincaré–Einstein metric. As mentioned above, the product of a Poincaré–Einstein metric with a sphere metric provides a large family of solutions to this system, which is known as Freund–Rubin solutions. More specifically, for any Poincaré–Einstein metric h with curvature $-6c^2$, the following metric and 4-form gives a solution to equations $Q(u) = 0$:

$$u = \left(h \times \frac{9}{c^2} g_{\mathbb{S}^4}, c dV_{\mathbb{S}^4}\right) \tag{2.7}$$

According to [15], there is a large class of Poincaré–Einstein metrics which can be obtained by perturbing the hyperbolic metric on the boundary. More specifically, there is the following result:

Theorem 2.1 ([6]). *Let $M = \mathbb{B}^{n+1}$ be the unit ball and $\hat{h}$ the standard metric on $\mathbb{S}^n$. For any smooth Riemannian metric $\hat{g}$ on $\mathbb{S}^n$ which is sufficiently close to $\hat{h}$ in $C^{2,\alpha}$norm if $n > 4$ or $C^{3,\alpha}$ norm if $n = 3$, for some $0 < \alpha < 1$, there exists a smooth metric g on the interior of M, with a C^0 conformal compactification satisfying*

$$\mathrm{Ric}(g) = -ng, \quad g \textit{ has conformal infinity } [\hat{g}].$$

We are mainly interested in the solutions that are perturbations of such a family of solutions, in particular, we will focus on the solutions with $c = 6$ above with hyperbolic metric on the ball and a scaled metric on the sphere, i.e. on $X = \mathbb{H}^7 \times \mathbb{S}^4$:

$$\left(g_H \times \frac{1}{4} g_S, 6 dV_{\mathbb{S}^4}\right), \tag{2.8}$$

which is also known as the Freund–Rubin solution.

2.1.4 Main theorem

In our theorem, we will fix a Poincaré–Einstein metric h on $\mathbb{B}^7$ which is sufficiently close to the hyperbolic metric. From the discussion above we know $(h \times \frac{1}{4} g_{\mathbb{S}^4}, 6dV_{\mathbb{S}^4})$ satisfy the gauged supergravity equation (2.4). As in Graham–Lee's paper, the Poincaré–Einstein metrics are paramtrized by the boundary data on $\mathbb{S}^6$, i.e. near any fixed Poincaré–Einstein metric, there exists a unique solution to the Einstein equation for a small (smooth) perturbation of the boundary conformal data.

For supergravity equations, we have additional parametrization data. Let

$$V_1 := \{v_1 \in C^\infty(\mathbb{S}^6; \bigwedge^3 T^*\mathbb{S}^6) \mid *_{\mathbb{S}^6} v_1 = iv_1\}.$$

Let $V_2^\pm$, $V_3^\pm$ be the smooth functions on the 6-sphere tensored with a finite dimensional 1-form space on $\mathbb{S}^4$:

$$V_2 := \{v_2 \otimes \xi_{16} : v_2 \in C^\infty(\mathbb{S}^6; \mathbb{R}), \xi_{16} \in E_{16}^{cl}(\mathbb{S}^4)\}$$

$$V_3 := \{v_3 \otimes \xi_{40} : v_3 \in C^\infty(\mathbb{S}^6; \mathbb{R}), \xi_{40} \in E_{40}^{cl}(\mathbb{S}^4)\}$$

where $E_{16}^{cl}(\mathbb{S}^4)$ and E_{40}^{cl} are closed 1-forms with eigenvalue 16 and 40 on the 4-sphere.

We also require three numbers that define the leading term in the expansion of the solution, which come from indicial roots:

$$\theta_1^\pm = 3 \pm 6i,\ \theta_2^\pm = 3 \pm i\sqrt{21116145}/1655,\ \theta_3 = 3 \pm i3\sqrt{582842}/20098. \tag{2.9}$$

If we fix an element $[\hat{h}]$ in the conformal boundary data to the leading order, the solution is parametrized by a small perturbation from the data on the bundle $C^\infty(\mathbb{S}^6; \oplus_{i=1}^3 V_i)$. The metric part of the solution to the leading order is given by the conformal infinity $[\hat{h}]$, whereas the form part to the leading order is given by the oscilatory data $v_i^+ x^{\theta_i^+} + S_i(v_i)x^{\theta^-}$.

To state the theorem, we give the following notations. Denote

$$u_0 := \left(g_{\mathbb{H}^7} \times \frac{1}{4} g_{\mathbb{S}^4}, 6\,\mathrm{Vol}_{\mathbb{S}^4}\right)$$

and $[\hat{h}]$ is close to $\hat{g}_0$ as metric on $\mathbb{S}^6$. A small neighborhood in the bundle is given by

$$U \subset C^\infty(\mathbb{S}^6; \oplus_{i=1}^3 V_i)$$

Theorem 2.2. *For $\delta \in (0,1)$, $s \geq 2$ and $k \gg 0$, in the space of solutions to $Q(u) = 0$*

in $x^{3-\delta}H_b^\infty(M;W)$*, a neighborhood of* u_0 *is smoothly parametrized by* $[\hat{h}]$ *and* U*. For a smooth section* $v \in U$ *with a sufficiently small* H^k *norm and a* $[\hat{h}]$*, there is a unique* $g \in x^{-\delta}H_b^\infty(M; \mathrm{Sym}^2({}^eT^*M))$ *and a 4-form* $F \in x^{-\delta}H_b^\infty(M; {}^e\bigwedge^4(T^*M))$ *prescribed by those data, such that* $(g-h, F-V_0) \in x^{-\delta}H_{e,b}^{s,k}(M;W)$ *and* $Q(u)=0$*.*

Our approach is based on the implicit function theorem. We consider the operator $Q_v(\cdot) = Q(\cdot + v)$. A right inverse of the linearization, denoted $(dQ_v)^{-1}$, is constructed, and we show that $Q_v \circ (dQ_v)^{-1}$ is an isomorphism on the Sobolev space $H_{e,b}^{s,k}(M;W)$ corresponding to the range of dQ_v.

To get the isomorphism result, we note that the model operator on the boundary is $SO(5)$-invariant, and therefore utilize the Hodge decomposition of functions and forms on S^4. We decompose the equations into blocks and compute the indicial roots of each block. The indicial roots are defined by the indical operator on each fiber $\pi_{-1}(p)$:

$$I_p[Q](s)v = x^{-s}Q(x^s v)|_{\pi^{-1}(p)}.$$

Indicial roots are those s that the indicial operator has a nontrivial kernel. These roots are related to the leading order of the solution expansions near the boundary.

Once the indicial roots are computed, we construct the operator $(dQ_v)^{-1}$. The operator exihibits different properties for large and small spherical eigenvalues. For large ones, the operator is already invertible by constructing a parametrix in the small edge calculus. For small eigenvalues, two resolvents $R_\pm = \lim_{\epsilon\to 0}(dQ \pm i\epsilon)^{-1}$ are constructed. We show that those elements corresponding to indicial roots with real part equal to 3 are the boundary perturbations needed in the theorem.

In section 2.2 we show the derivation of the equations from the Lagrangian and discuss the gauge breaking condition. In section 2.3 we study the linearized operator and show it is Fredholm on suitable edge Sobolev spaces. In section 2.4 we construct the solutions for the nonlinear equations using the implicit function theorem.

2.2 Gauged operator construction

2.2.1 Equations derived from Lagrangian

The supergravity system is derived as the variational equations for the following Lagrangian:

$$L(g, A) = \int_X R dV_g - \frac{1}{2}\Big(\int_X F \wedge *F + \int_X \frac{1}{3} A \wedge F \wedge F\Big).$$

Now we compute its variation along two directions, namely, the metric and the form direction. The first term is the Einstein-Hilbert action, for which the variation in g is

$$\delta_g\Big(\int dV_g R\Big) = \int \Big(R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} \delta g^{\alpha\beta} dV_g\Big). \tag{2.10}$$

Now we compute the variation of the second term $F \wedge *F$ in the metric direction, which is

$$\delta_g\Big(\frac{1}{2}\int F \wedge *F\Big) = \frac{2}{4!}\int F_{\eta_1 \ldots \eta_4} F_{\xi_1 \ldots \xi_4} g^{\eta_2 \xi_2} g^{\eta_3 \xi_3} g^{\eta_4 \xi_4} \delta g^{\eta_1 \xi_1} dV_g - \frac{1}{4}\int F \wedge *F g_{\alpha\beta} \delta g^{\alpha\beta}. \tag{2.11}$$

Combining the two variations and setting them equal to zero, we get

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{1}{12} F_{\alpha\eta_1\eta_2\eta_3} F_\beta^{\eta_1\eta_2\eta_3} - \frac{1}{4}\langle F, F\rangle g_{\alpha\beta}. \tag{2.12}$$

Here $\langle \bullet, \bullet \rangle$ is the inner product on forms:

$$\langle F, F\rangle = \frac{1}{4!} F_{\eta_1 \ldots \eta_4} F^{\eta_1 \ldots \eta_4}. \tag{2.13}$$

Taking the trace of the equation, we get

$$R = \frac{1}{6}\langle F, F\rangle. \tag{2.14}$$

Finally, substituting R in the equation, we get

$$R_{\alpha\beta} = \frac{1}{12}(F_{\alpha\gamma_1\gamma_2\gamma_3}F_{\beta}^{\gamma_1\gamma_2\gamma_3} - \frac{1}{12}F_{\gamma_1\gamma_2\gamma_3\gamma_4}F^{\gamma_1\gamma_2\gamma_3\gamma_4}g_{\alpha\beta}), \tag{2.15}$$

which gives the first equation in supergravity system2.1.

The variation with respect to the 3-form A is

$$\delta_A S = \int \delta F \wedge * F - \frac{1}{6}\delta A \wedge F \wedge F - \frac{1}{3}\int A \wedge \delta F \wedge F = -\int \delta A \wedge (d * F + \frac{1}{2} F \wedge F), \tag{2.16}$$

which gives the second supergravity equation:

$$d * F + \frac{1}{2} F \wedge F = 0. \tag{2.17}$$

Since F is the differential of A, so we have the third equation

$$dF = 0. \tag{2.18}$$

Product solutions are obtained as follows: let X^7 be an Einstein manifold with negative scalar curvature $\alpha < 0$ and K^4 be an Einstein manifold with positive scalar curvature $\beta > 0$. Consider $X \times K$ with the product metric; then we have

$$Ric = \begin{pmatrix} 6\alpha g^M_{AB} & 0 \\ 0 & 3\beta g^K_{ab} \end{pmatrix} \tag{2.19}$$

Let $F = cV_K$. A straightforward computation shows

$$(F \circ F)_{\alpha\beta} = \frac{c^2}{12}\begin{pmatrix} -2g^M_{AB} & 0 \\ 0 & 4g^K_{ab} \end{pmatrix}. \tag{2.20}$$

Therefore any set (c, α, β) satisfying

$$-c^2/6 = 6\alpha, c^2/3 = 3\beta \tag{2.21}$$

corresponds to a solution to the supergravity equation.

2.2.2 Change to a square system

In order to write the equations as a square system, we change the second equation to second order by applying $d*$. Combining with the closed condition $dF = 0$, $d * d * F$ is the same as $\varDelta F$. This leads to the following square system

$$\operatorname{Ric} g - F \circ F = 0 \tag{2.22}$$

$$\varDelta F + \frac{1}{2} d * (F \wedge F) = 0 \tag{2.23}$$

Proposition 2.1. *After changing to the square system, the kernel is the same as the kernel of the original system.*

Proof. The only change here is that we introduced $d* = *_M (d_{\mathbb{S}^4} + d_{\mathbb{H}^7}) *_M$, in which $*_M$ and $d_{\mathbb{S}^4}$ are both isomorphisms. Therefore we only need to consider the possible kernel introduced by $d_{\mathbb{H}^7}$ in the solution space. On the hyperpolic space, there are no L^2 kernel for $d_{\mathbb{H}^7}$ because of the representation, and the nonlinear terms after taking off the kernel of the linearized operatoris of decay $x^{3+\delta}$. □

2.2.3 Gauge condition

Following [15] in the setting of Poincaré–Einstein metric, we add a gauge operator to the curvature term where g is the background metric:

$$\phi(t, g) = \delta_t^* (tg)^{-1} \delta_t G_t g.$$

Here

$$[G_t g]_{ij} = g_{ij} - \frac{1}{2} g_k^k t_{ij}, \quad [\delta_t g]_i = -g_{ij,}^j$$

δ_g^* is the formal adjoint of δ_g, which can be written as

$$[\delta_g^* w]_{ij} = \frac{1}{2}(w_{i,j} + w_{j,i}),$$

and

$$[(tg)^{-1}w]_i = t_{ij}(g^{-1})^{jk}w_k.$$

By adding the gauge term we get an operator Q, which is a map from the space of symmetric 2-tensors and closed 4-forms to the space of symmetric 2-tensors and closed 4-forms:

$$Q : S^2(T^*M) \oplus \bigwedge\nolimits_{cl}^4(M) \to S^2(T^*M) \oplus \bigwedge\nolimits_{cl}^4(M)$$

$$\begin{pmatrix} g \\ F \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{Ric}(g) - \phi(t,g) - F \circ F \\ d * (d * F + \frac{1}{2} F \wedge F) \end{pmatrix} \tag{2.24}$$

which will be the main object to study.

As discussed in [15], $Ric(g) + ng - \phi(t,g) = 0$ holds if and only if $id : (M,g) \to (M,t)$ is harmonic and $Ric(g) + ng = 0$. We will show that the gauged equations here yield the solution to the supergravity equations in a similar manner.

We first prove a gauge elimination lemma for the linearized operator. As can be seen from (2.24), only the first part (the map on 2-tensors) involves the gauge term, therefore we restrict the discussion to the first part of dQ. We use $dQ_g(k,H)$ to denote the linearization of the tensor part of Q along the metric direction at the point (g,F), which acts on (k,H). First we give the following gauge-breaking lemma for the linearized operator, which is adapted from Theorem 4.2 in [20].

Lemma 2.1. *If (k,H) satisfies the linearized equation $dQ_g(k,H) = 0$, then there exists a 1-form v and $\tilde{k} = k + L_{v^\sharp} g$ such that $dS_g(\tilde{k},H) = 0$.*

To prove the proposition, we first determine the equation to solve for such a 1-form v.

Lemma 2.2. *If a 1-form v satisfies*

$$((\Delta^{rough} - Ric)v)_\lambda = \frac{1}{2}(2\nabla^\alpha k_{\alpha\lambda} - \nabla_\lambda Tr_g(k)) \tag{2.25}$$

then $\tilde{k} = k + L_{v^\sharp} g$ *satisfies the gauge condition*

$$d\phi(t,g)_g(\tilde{k}) = 0.$$

.

Proof. Let $\Psi(v,g)$ be the map

$$\Psi(v,g)^k = (\phi^*_{v^\sharp} g)^{\alpha\beta}(\Gamma^k_{\alpha\beta}(\phi^*_{v^\sharp} g) - \Gamma^k_{\alpha\beta}(t)).$$

This satisfies

$$\delta^*_g g D_v \Psi(0,t)(v) = D_t\phi(L_{v^\sharp t} g), \delta^*_g g D_g \Psi(0,t)(k) = D_t\phi(k).$$

Therefore in order to get $D_t\phi(\tilde{k}) = 0$, we only need

$$-D_v\Psi(0,t)(v) = D_g\Psi(0,t)(k).$$

The left hand side can be reduced to

$$-g^{\alpha\beta}\nabla_\alpha\nabla_\beta v^k - R^k_\mu v^\mu = (\Delta^{rough} - Ric)v^k$$

and right hand side is

$$\frac{1}{2} g^{\alpha\beta} g^{k\lambda}(\nabla_\alpha k_{\beta\lambda} + \nabla_\beta k_{\alpha\lambda} - \nabla_\lambda k_{\alpha\beta}).$$

Lowering the index on both side, we get

$$((\Delta^{rough} - Ric)v)_\lambda = \frac{1}{2}(2\nabla^\alpha k_{\alpha\lambda} - \nabla_\lambda Tr_g(k)).$$

□

Next we discuss the solvability of the operator defined above in (2.25).

Lemma 2.3. *If $|\delta| < 1$, then at the point $g_0 = g_H \times \frac{1}{4} g_S$ the operator*

$$\mathit{\Delta}^{rough} - Ric : x^\delta H^2({}^eT^*M) \to x^\delta L^2({}^eT^*M)$$

is an isomorphism.

Proof. Using the splitting

$${}^eT^*M \cong \pi_{\mathbb{H}}^{*}\,{}^eT^*\mathbb{H}^7 \oplus \pi_{\mathbb{S}}^{*e}T^*\mathbb{S}^4$$

and the product structure of the metric, we write the operator as

$$\mathit{\Delta}^{rough} - Ric = \mathit{\Delta}_H^{rough} + \mathit{\Delta}_S^{rough} - \mathrm{diag}(-6, 12).$$

It decomposes into two parts: trace and trace-free 2-tensors. Decomposing into eigenfunctions on the 4-sphere, consider the following two operators:

$$L_{tr} = \mathit{\Delta}_{\mathbb{H}} + \lambda - 24 : C^\infty(\mathbb{H}^7) \to C^\infty(\mathbb{H}^7), \tag{2.26}$$

$$L_{tf} = \mathit{\Delta}_{\mathbb{H}}^{rough} + \lambda' + 6 : {}^e\bigwedge^* \mathbb{H}^7 \to {}^e\bigwedge^* \mathbb{H}^7 \tag{2.27}$$

Consider the smallest eigenvalue in each case: $\lambda_{tr} = 16, \lambda_{tf} = 0$. The indicial radius for L_{tr} is 1 and for L_{tf} is 4. Then using theorem 6.1 in [25], for $|\delta| < 1$ the operator L_{tr} as a map from $x^\delta H^2(\mathbb{H}^7) \to x^\delta L^2(\mathbb{H}^7)$ has closed range and is essentially injective. Moreover, since the kernel of this operator lies in the L^2 eigen space of $\mathbb{H}^7$ which vanishes, therefore the operator is actually injective. By self-adjointness, it is also surjective on the same range for δ. The same argument holds for L_{tf}, which is also Fredholm and an isomorphism on $x^\delta H^2({}^e\bigwedge^* \mathbb{H}^7) \to x^\delta L^2({}^e\bigwedge^* \mathbb{H}^7)$ for $|\delta| < 4$.

Combining the statements for L_{tr} and L_{tf}, we conclude that $\mathit{\Delta}^{rough} - Ric$ is an isomorphism between $x^\delta H^2({}^eT^*M) \to x^\delta L^2({}^eT^*M)$ for $|\delta| < 1$.

□

The isomorphism holds true for metrics nearby, by a simple perturbation argument.

Corollary 1. *At any metric g which is close to g_0, for $|\delta| < 1$, $\Delta^{rough} - Ric$ is an isomorphism as a map*

$$\Delta^{rough} - Ric : x^\delta H^2({}^eT^*M) \to x^\delta L^2({}^eT^*M).$$

With the lemmas above, we can prove the proposition.

Proof of Proposition 2.1. From Lemma 1 we know $\Delta^{rough} - Ric : x^\delta H^2({}^eT^*M) \to x^\delta L^2({}^eT^*M)$ is an isomorphism, therefore there exists a one-form v satisfying 2.2. Then from Lemma 2.2, $\tilde{k} = k + L_{v^\sharp} g$ satisfies $D_g\phi(\tilde{k}) = 0$. Putting it back to the linearized equation, we get $dS_g(\tilde{k}, H) = 0$. □

Next we prove the nonlinear version of gauge elimination by using integral curves.

Proposition 2.2. *If a metric and a closed 4-form (g, V) satisfies the gauged equations $Q(g, V) = 0$, then there is an diffeomorphism $g \mapsto \tilde{g}$ such that $\phi(\tilde{g}, t) = 0$ and $(\tilde{g}, V)$ is a solution to equation(2.1) i.e. $S(\tilde{g}, V) = 0$.*

Proof. Consider the integral curve on the manifold of metrics close to the product metric, defined by the vector field at each point g with value $\tilde{k}_g = k_g + L_{v^\sharp} g$. Then along this curve, it satisfies $D_g Q(\tilde{k}_g, H) = 0$. Therefore $Q(\tilde{g}, V) = 0$.

We also need to show that $\tilde{g}$ has the same regularity as g. It suffices to show that the vector field is smooth enough. Since v solved above in Lemma 2.2 is polyhomogeneous, then the Lie derivative $L_{v^\sharp} g = \nabla_\alpha v_\beta + \nabla_\beta v_\alpha$ is one order less smooth than v. However by integration we gain one order of regularity back. Therefore $\tilde{g}$ has the same regularity as g. □

2.3 Fredholm property of the linearized operator

We now consider the linearization of the gauged supergravity operator near the base metric and 4-form $(g_0, F) = (g_{\mathbb{H}^7} \times \frac{1}{4} g_{\mathbb{S}^4}, 6\,\mathrm{Vol}_{\mathbb{S}^4})$. The first step is to compute the indicial roots and indicial kernels of this linearizaed operator, which is done with respect of the harmonic decomposition of the 4-sphere. As the eigenvalues becomes larger, the indicial roots become further apart. Specifically, all of them are separated by real part 3, with only three pairs of exceptions, where the indicial roots corresponding to the lowest three eigenvalues lie on the L^2 line which has real part equal to 3.

Once we identify these indicial roots, we proceed differently according to whether the indicial roots land on the L^2 line or not. We show that for most of the indicial roots, the decomposed linearized operator is Fredholm on suitable edge Sobolev spaces. This is done by using small edge calculus and $SO(5)$ invariance of the structure. For the three exceptional pairs we use scattering theory to construct two generalized inverses, which encode the boundary data that parametrize the kernel of the linear operator.

We then describe the kernel of this linearized operator in terms of the two generalized inverses, and a scattering matrix construction that gives the Poisson operator. Nearby the Poincaré–Einstein metric product, a perturbation argument shows that the space given by the difference of the two generalized inverses is transversal to the range space of the linearized operator and therefore this space gives the kernel of the linearized operator, which later will provide the kernel parametrization for the nonlinear operator.

2.3.1 Linearization of the operator Q

The nonlinear supergravity operator contains two parts: the gauged curvature operator $\mathrm{Ric} - \phi_{g,t}$ with its nonlinear part $F \circ F$, and the first order differential of the 4-form $d * F$ with its nonlinear part $F \wedge F$. Note that since the Hodge operator $*$ depends on the metric, the linearized operator couples the metric and the 4-form in both of the equations.

Though we only consider the linearization about the base poduct metric $g_{\mathbb{H}^7} \times \frac{1}{4} g_{\mathbb{S}^4}$, the computation below applies to other Poincaré–Einstein metrics satisfying the relation 2.7 mentioned in the introduction. Since near the boundary the metric is the same as $\mathbb{H}^7 \times \mathbb{S}^4$, the discussion about edge operators in later sections would remain the same.

Proposition 2.3. *The operator $Q : W \to W$ has the following linearization at the point (g_0, F):*

$$dQ_{g_0,F} : \Gamma(\mathrm{Sym}^2({}^eT^*M) \oplus {}^e\bigwedge^4(M)) \to \Gamma(\mathrm{Sym}^2({}^eT^*M) \oplus {}^e\bigwedge^4(M))$$

$$\begin{pmatrix} k \\ H \end{pmatrix} \mapsto \begin{pmatrix} \Delta k + \mathrm{LOT} \\ d * (d * H + 6\,\mathrm{Vol}_{\mathbb{S}} \wedge H + 6d *_{\mathbb{H}} k_{1,1} + 3d(\mathrm{tr}_{\mathbb{H}^7}(k) - \mathrm{tr}_{\mathbb{S}^4}(k)) \wedge \mathrm{Vol}_{\mathbb{H}}) \end{pmatrix} \tag{2.28}$$

where the lower order term matrix LOT *is as follows:*

$$\mathrm{LOT} = \begin{pmatrix} -k_{IJ} - 6\,\mathrm{tr}_{\mathbb{S}}(k)t_{IJ} + \mathrm{tr}_{\mathbb{H}}(k)t_{IJ} + 2 *_{\mathbb{S}} H_{0,4}t_{IJ} & 6k_{1,1} - 3 *_{\mathbb{S}} H_{1,3} \\ 6k_{1,1} - 3 *_{\mathbb{S}} H_{1,3} & 4k_{ij} + 8\,\mathrm{tr}_{\mathbb{S}}(k)t_{ij} - *_{\mathbb{S}}H_{0,4}t_{ij} \end{pmatrix}$$

We break the computation into a curvature part and a form part as follows.

Lemma 2.4. *For $k \in \mathrm{Sym}^2({}^eT^*M)$, the linearization of the gauge broken Ricci operator at the base hyperbolic metric is*

$$d_{g_0}(\mathrm{Ric} - \phi_{g,t})(k) = \frac{1}{2}\Delta^{rough}_{g_0} k + R(k), \tag{2.29}$$

where

$$R(k) = \begin{pmatrix} -7k_{IJ} + \mathrm{Tr}_{\mathbb{H}^7}(k)g_{IJ} & 0 \\ 0 & 16k_{ij} - \mathrm{Tr}_{\mathbb{S}^4}(k)g_{ij} \end{pmatrix}.$$

Proof. Following the result in [15], the linearization of the gauged operator at the base metric t is

$$d_t(\mathrm{Ric} - \phi_{g,t})(k) = \frac{1}{2}\Delta^{rough}_t k + k^{\alpha\beta}R_{\beta\gamma\delta\alpha} + \frac{1}{2}(R^{\beta}_{\gamma}k_{\beta\delta} + R^{\beta}_{\delta}k_{\beta\gamma}). \tag{2.30}$$

Specifically, if the metric is constant sectional curvature near the boundary of M which is the case for the conformal compact metric here (with sectional curvature -1), the curvature term is diagonalized and can be written as

$$R_{\alpha\beta\delta\beta} = -(g_{\alpha\delta}g_{\gamma\beta} - g_{\alpha\beta}g_{\gamma\delta}),$$

so the linearization of this total operator is as above. □

Lemma 2.5. *The linearization of $F \circ F$ along the 2-tensor direction and 4-form direction are: for $k \in \mathrm{Sym}^2({}^eT^*M)$*

$$d_{g_0,6\,\mathrm{Vol}_{\mathbb{S}}}(F\circ F)(k) = \begin{pmatrix} \frac{1}{36}Tr_S(k)t_{IJ} - \frac{1}{36}k + 2\langle W, H\rangle t & \frac{1}{144}\langle W,W\rangle k_{Ij} \\ \frac{1}{144}\langle W,W\rangle k_{Ij} & \begin{matrix} \frac{1}{12}\big(-3W_{ai_1i_2i_3}W^{j_2j_3}_{bj_1}t^{i_1l_1}k_{l_1l_2}t^{l_2j_1} \\ +\frac{4}{12}F_{i_1i_2i_3i_4}F^{i_1i_2i_3}_{j_1}t^{j_1l_1}k_{l_1l_2}t^{l_2i_1}t_{ab} - \frac{1}{12}\langle W,W\rangle k_{ab}\big)\end{matrix} \end{pmatrix}, \tag{2.31}$$

and for $H \in {}^e\bigwedge^4(T^*M)$

$$d_{g_0,6\,\mathrm{Vol}_{\mathbb{S}}}(F\circ F)(H) = \begin{pmatrix} \frac{1}{72}c_2 *_S H_{(0,4)}t_{AB} & \frac{1}{3}(*_S H_{(1,3)})_{Ab} \\ 3(*_S H_{(1,3)})_{Ab} & \begin{matrix}\frac{1}{6}H_a^{i_1i_2i_3}W_{bi_1i_2i_3} \\ -\frac{1}{72}W_{i_1i_2i_3i_4}H^{i_1i_2i_3i_4}t_{ab}\end{matrix} \end{pmatrix}. \tag{2.32}$$

Proof. The proof is by direct computation. Note that

$$D_{t,W}(F\circ F)(H) = \begin{pmatrix} HH & HS \\ HS & SS \end{pmatrix},$$

where

$$HH_{AB} = -\frac{1}{72}W_{i_1i_2i_3i_4}H^{i_1i_2i_3i_4}t_{AB} = -\frac{1}{72}c_2 h t_{AB} = \frac{1}{72}c_2 *_{\mathbb{S}} H_{(0,4)}t_{AB}; \tag{2.33}$$

$((\mathrm{Vol}_{\mathbb{S}})_{i_1i_2i_3i_4}(V_S)^{i_1i_2i_3i_4} = 1, *_{\mathbb{S}}\mathrm{Vol}_{\mathbb{S}} = 1, H_{(0,4)} = h\,\mathrm{Vol}_{\mathbb{S}}.)$

$$HS_{Ab} = \frac{1}{12}H_{Ai_1i_2i_3}W_b^{i_1i_2i_3} = 3(*_{\mathbb{S}}H_{(1,3)})_{Ab}, \tag{2.34}$$

$$SS = \frac{1}{6}H_a^{i_1i_2i_3}W_{bi_1i_2i_3} - \frac{1}{72}W_{i_1i_2i_3i_4}H^{i_1i_2i_3i_4}t_{ab}. \tag{2.35}$$

Then for metric variation $k \in \mathrm{Sym}^2(T^*M)$

$$D_{t,W}(F \circ F)(k) = \begin{pmatrix} HH & HS \\ HS & SS \end{pmatrix},$$

$$HS = \frac{1}{144} W_{i_1 i_2 i_3 i_4} W^{i_1 i_2 i_3 i_4} k_{Ij}, \tag{2.36}$$

$$HH_{AB} = \frac{1}{12}\Big(\frac{4}{12} F_{i_1 i_2 i_3 i_4} F^{i_1 i_2 i_3}_{j_1} t^{j_1 l_1} k_{l_1 l_2} t^{l_2 i_1} t_{AB} - \frac{1}{12} W_{1_1 i_2 i_3 i_4} W^{i_1 i_2 i_3 i_4} k_{AB}\Big), \tag{2.37}$$

$$\begin{aligned} SS_{ab} = \frac{1}{12}\Big(- 3W_{a i_1 i_2 i_3} W^{j_2 j_3}_{b j_1} t^{i_1 l_1} k_{l_1 l_2} t^{l_2 j_1} + \frac{4}{12} F_{i_1 i_2 i_3 i_4} F^{i_1 i_2 i_3}_{j_1} t^{j_1 l_1} k_{l_1 l_2} t^{l_2 i_1} t_{ab} \\ - \frac{1}{12} W_{i_1 i_2 i_3 i_4} W^{i_1 i_2 i_3 i_4} k_{ab}\Big), \end{aligned} \tag{2.38}$$

which using inner product $W_{i_1 i_2 i_3 i_4} W^{i_1 i_2 i_3 i_4} = \langle W, W \rangle$ will give the expressions above.

□

Next we compute the linearization of the 2nd equation:

Lemma 2.6. *The linearization of the equation*

$$d * F + \frac{1}{2} F \wedge F = 0,$$

along the form direction and tensor direction are respectively:

$$d_{g_0, F_0}(d * F + \frac{1}{2} F \wedge F)(H) = d * H + H \wedge F, \tag{2.39}$$

$$d_{g_0, F_0}(d * F + \frac{1}{2} F \wedge F)(k) = 6 d *_{\mathbb{H}} k_{(1,1)} + 3d(\mathrm{tr}_{\mathbb{H}}(k) - \mathrm{tr}_{\mathbb{S}}(k)) \,\mathrm{Vol}_{\mathbb{H}} \tag{2.40}$$

Proof. The linearization along the form direction is straight-forward, as the terms are linear and quadratic on F. Along the metric direction, the linearization comes from the Hodge star:

$$
\begin{aligned}
&D(*F)_{\beta_1\beta_2..\beta_7}(k)\\
&= D(\tfrac{1}{4!}V^{\alpha_1..\alpha_4}_{\beta_1..\beta_7}W_{\alpha_1..\alpha_4})(k) = \tfrac{1}{4!}(\delta V)^{\alpha_1..\alpha_4}_{\beta_1..\beta_7}W_{\alpha_1..\alpha_4} + \tfrac{4}{4!}V^{\alpha_2..\alpha_4}_{\gamma_1\beta_1..\beta_7}(\delta g)^{\gamma_1\alpha_1}W_{\alpha_1..\alpha_4}\\
&= \tfrac{1}{2}\tfrac{1}{4!}t^{\alpha\beta}k_{\alpha\beta}V^{\alpha_1..\alpha_4}_{\beta_1..\beta_7}W_{\alpha_1..\alpha_4} + \tfrac{1}{6}V^{\alpha_2..\alpha_4}_{\gamma_1\beta_1..\beta_7}t^{\gamma_1\xi}k_{\xi\psi}t^{\psi\alpha_1}W_{\alpha_1..\alpha_4}\\
&= 6d *_{\mathbb{H}} k_{(1,1)} + 3d(\mathrm{tr}_{\mathbb{H}}(k) - \mathrm{tr}_{\mathbb{S}}(k))\,\mathrm{Vol}_{\mathbb{H}}
\end{aligned}
\tag{2.41}
$$

which gives the expressions above. □

Proof of Propsition 2.3. Combining everything together, the linearized equations are

$$
\begin{aligned}
&d * (6d *_{\mathbb{H}} k_{(1,1)} + 3d(\mathrm{tr}_{\mathbb{H}}(k) - \mathrm{tr}_{\mathbb{S}}(k))\,\mathrm{Vol}_{\mathbb{H}} + d * H + H \wedge \mathrm{Vol}_b\, bH) = 0\\
&\tfrac{1}{2}\Delta^{rough}_{g_0}k + k^{\alpha\beta}R_{\beta\gamma\delta\alpha} + \tfrac{1}{2}(R^{\beta}_{\gamma}k_{\beta\delta} + R^{\beta}_{\delta}k_{\beta\gamma}) + LOT = 0
\end{aligned}
\tag{2.42}
$$

which after arrangement gives the linearization in (2.28). □

2.3.2 Indicial roots computation

Having obtained the linearized operator dQ, we next compute its indicial roots on the boundary of $\mathbb{H}^7$, which together with its indicial kernels parametrize the boundary values of this linear operator. Utilizing the Hodge decomposition on the 4-sphere, the operator acts on a space of sections on $\mathbb{H}^7$ tensored with finite dimensional subspaces of $\Gamma(T^*\mathbb{S}^4)$.

Definition 2.1 (Hodge decomposition projection)**.** *Let λ be one of the eigenvalues for the Hodge laplacian on $\Gamma(\oplus_{i=0}^4 \bigwedge^i(T^*\mathbb{S}^4))$ and define the eigenvalue projection operator π_λ on the the sections of the bundle $W = \mathrm{Sym}^2({}^eT^*M) \oplus {}^4\bigwedge {}^eT^*M$ to be the projection that maps to the corresponding part on sphere.*

Note here we have a collection of eigenvalues on both functions and forms.

Lemma 2.7. *Sections of the bundle W decompose according to eigenvalues λ.*

Proof. We identify the symmetric edge 2-tensor bundle with $(\mathrm{Sym}^2({}^eT^*\mathbb{H}^7)) \oplus ({}^eT^*\mathbb{H}^7 \otimes T^*\mathbb{S}^4) \oplus \mathrm{Sym}^2(T^*\mathbb{S}^4)$, and decompose the 4-form bundle according to its degree on

$\mathbb{H}^7$ and $\mathbb{S}^4$, i.e. ${}^e\bigwedge^4 T^*M = \oplus_{i+j=4} {}^e\bigwedge^i T^*\mathbb{H}^7 \otimes \bigwedge^j T^*\mathbb{S}^4$. And for each element of the form $u \otimes v$ with $u \in \Gamma({}^e\bigwedge^* T^*\mathbb{H}^7), v \in \Gamma(\bigwedge^* T^*\mathbb{S}^4)$ the projection operator π_λ maps it to $u \otimes \pi_\lambda v$, which by linearity extends to the whole bundle W. □

It follows that the operator decomposes to an infinite collection of operators, each acting on a subbundle.

Lemma 2.8. *The operator dQ preserves the eigenspaces of $\mathbb{S}^4$, and we have the following decomposition of the operator*

$$dQ = \sum_{\lambda \geq 0} dQ^\lambda := \sum_\lambda \pi_\lambda \circ dQ \circ \pi_\lambda$$

Proof. We only need to show that that Hodge laplacian Δ commutes with the linearized operator. Since the linear operator is composed from Δ^{hodge}, Δ^{rough} (which are related by Bochner formula), Hodge $*$ operator, differential, and scalar operator, all of which commute with Δ, dQ therefore commutes with the eigenvalue projections. □

Now we define indicial roots and indicial kernels below for the edge operator. Recall that ∂M is the total space of fibration over $Y = \partial\mathbb{B}^7$.

Definition 2.2 (Indicial operator)**.** *Let $L : \Gamma(E_1) \to \Gamma(E_2)$ be an edge operator between two vector bundles over M. For any boundary point $p \in Y$, and $s \in \mathbb{C}$, the indicial operator of L at point p is defined as*

$$I_p[L](s) : \Gamma(E_1|_{\pi^{-1}(p)}) \to \Gamma(E_2|_{\pi^{-1}(p)})$$

$$(I_p[L](s))v = x^{-s} L(x^s \tilde{v})|_{\pi^{-1}(p)}$$

where $\tilde{v}$ is an extension of v to a neighborhood of $\pi^{-1}(p)$. The indicial roots of L at point p are those $s \in \mathbb{C}$ such that $I_p[L](s)$ has a nontrivial kernel, and the corresponding kernels are called indicial kernels.

In the conformally compact case, the indicial operator is a bundle map from $E_1|_p$ to $E_2|_p$ (which is simpler than a partial differential operator as in the general edge

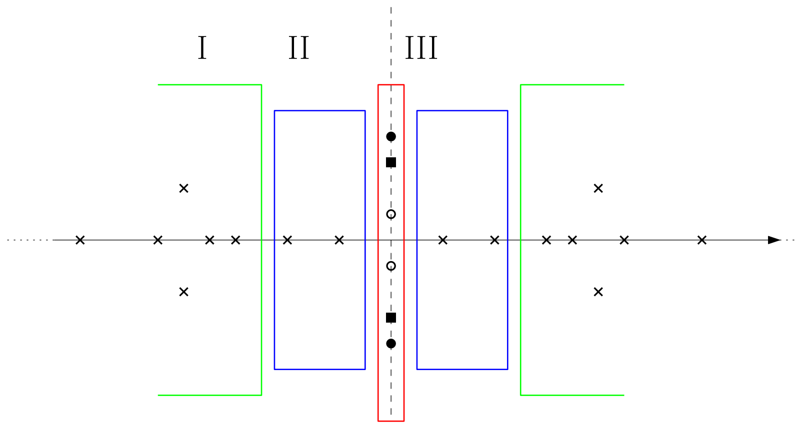

Figure 2-1: Indicial roots of the linearized supergravity operator on $\mathbb{C}$

case). Moreover, since we have an $SO(7)$ symmetry for the operator, the indicial roots will be constants for any boundary point $p \in \mathbb{S}^6$.

Proposition 2.4. *The indicial roots of operator dQ are symmetric around $\operatorname{Re} z = 0$, with three special pairs of roots*

$$\theta_1^{\pm} = 3 \pm 6i,\ \theta_2^{\pm} = 3 \pm i\sqrt{21116145}/1655,\ \theta_3 = 3 \pm i3\sqrt{582842}/20098.$$

and all other roots lying in $\{\|\operatorname{Re} z - 3\| \geq 1\}$.

Proof. With the harmonic decomposition on sphere $\mathbb{S}^4$, the linearized operator dQ is block-diagonalized and we compute the indicial roots for the linear system dQ in Section 2.6. We summarize the results below and Figure 2-1 is an illustration of the indicial roots distribution. The indicial roots fall into the following three categories:

1. The roots corresponding to harmonic forms:

   (a) The equation for trace-free 2-tensors on H^7 arising from the first component of (2.28) is

   $$(\varDelta_S + \varDelta_H - 2)\hat{k}_{IJ} = 0,$$

   and the corresponding indicial equation is

   $$(-s^2 + 6s)k_{IJ} = 0.$$

We have indicial roots

$$S_1^+ = 0, S_1^- = 6.$$

This corresponds to the perturbation of the hyperbolic metric to a Poincaré–Einstein metric.

(b) The equation for trace-free 2-tensors on S^4 is

$$\Delta_S^{rough}\hat{k}_{ij} + \Delta_H\hat{k}_{ij} + 8\hat{k}_{ij} = 0$$

where indicial equation is

$$(-s^2 + 6s + 8)\hat{k}_{ij} = 0,$$

and the indicial roots are

$$S_2^{\pm} = 3 \pm \sqrt{17}.$$

(c) Equations for $H_{(4,0)}$:

$$d_H * H_{(4,0)} + W \wedge H_{(4,0)} = 0$$
$$d_H H_{(4,0)} = 0$$

where the indicial equation is

$$-(s-3)(*_6 N) \wedge dx/x - 6dx/x \wedge N = 0,$$

with indicial roots

$$\theta_1^{\pm} = 3 \pm 6i.$$

This corresponds to a perturbation of the 4-form on hyperbolic space.

2. The roots corresponding to functions / closed 1-forms / coclosed 3-forms / closed 4-forms

(a) The equations for $7\sigma = Tr_H(k), 4\tau = Tr_S(k), k_{(1,1)}, H_{(1,3)}, H_{(0,4)}$ are

$$6d_H *_H k^{cl}_{(1,1)} + d_S(3Tr_H(k) - 3Tr_S(k)) \wedge {}^7V + d_S * H^{cl}_{(0,4)} + d_H * H^{cc}_{(1,3)} = 0$$
$$d_H H^{cl}_{(0,4)} + d_S H^{cc}_{(1,3)} = 0$$
$$d_H H^{cc}_{(1,3)} = 0$$
$$\triangle_s k^{cl}_{(1,1)} + \triangle_H k^{cl}_{(1,1)} + 12k^{cl}_{(1,1)} - 6 *_S H^{cc}_{(1,3)} = 0$$
$$\Delta_S \tau + \Delta_H \tau + 72\tau - 8 *_S H^{cl}_{0,4} = 0$$
$$\Delta_S \sigma + \Delta_H \sigma + 12\sigma + 4 *_S H^{cl}_{0,4} - 48\tau = 0$$

The indicial equations are

$$\begin{aligned} &\lambda^4 - 4S^2\lambda^3 + 24S * \lambda^3 - 90\lambda^3 + 6S^4\lambda^2 - 72S^3\lambda^2 \\ &+ 342S^2\lambda^2 - 756S * \lambda^2 + 1152\lambda^2 - 4S^6\lambda + 72S^5\lambda - 414S^4\lambda \\ &+ 648S^3\lambda + 1152S^2\lambda - 3024S * \lambda + 10368\lambda \\ &+ S^8 - 24S^7 + 162S^6 + 108S^5 - 6192S^4 + 31536S^3 - 33696S^2 - 155520S = 0 \end{aligned} \tag{2.43}$$

When $\lambda = 16$ there is a pair of roots with real part 3

$$s = \theta_2^{\pm} = 3 \pm i\sqrt{21116145}/1655 \tag{2.44}$$

and when $\lambda = 40$ there is a pair of roots with real part 3

$$\theta_3^{\pm} = 3 \pm i3\sqrt{582842}/20098$$

And here the five variables are related by

$$H^{cl}_{(0,4)} = d_s *_s d_s\xi, H^{cc}_{(1,3)} = -d_H *_s d_s\xi, k^{cl}_{(1,1)} = -d_s\delta_H\xi, 4\sigma = 7\tau = \xi$$

where

$$\xi \in \delta_s \wedge^1_{16} S^4$$

similarly we have another indicial kernel corresponding to $\theta_3^{\pm}$ with $\xi \in \delta_s \wedge^1_{40} S^4$.

(b) The equations for $H_{(3,1)}, H_{(4,0)}$ are

$$
\begin{aligned}
&d_S * H^{cl}_{(3,1)} + d_H * H^{cc}_{(4,0)} + 6^4 V \wedge H^{cc}_{(4,0)} = 0 \\
&d_H H^{cl}_{(3,1)} + d_S H^{cc}_{(4,0)} = 0
\end{aligned}
$$

where the indicial equations are

$$
(s-3)^2 \pm 6i(s-3) - 16 = 0
$$

with indicial roots

$$
S_6^{\pm} = 3 \pm \sqrt{7} \pm 3i.
$$

3. The roots corresponding to coclosed 1-forms / closed 2-forms / coclosed 2-forms / closed 3-forms

(a) The equations for $k_{(1,1)}, H_{(1,3)}, H_{(2,2)}$ are

$$
\begin{aligned}
&6 d_H *_H k^{cc}_{(1,1)} + d_H * H^{cl}_{(1,3)} = 0 \\
&d_S * H^{cl}_{(1,3)} + d_H * H^{cc}_{(2,2)} + 6 d_S *_H k^{cc}_{(1,1)} = 0 \\
&d_H H^{cl}_{(1,3)} + d_S H^{cc}_{(2,2)} = 0 \\
&\tfrac{1}{2} \triangle_s k^{cc}_{(1,1)} + \tfrac{1}{2} \triangle_H k^{cc}_{(1,1)} + 6 k^{cc}_{(1,1)} - \tfrac{1}{2} *_S H^{cl}_{(1,3)} = 0
\end{aligned}
$$

The indicial equation is

$$
\lambda^2 - (36 + (s-1)(s-5) + s^2 - 6s - 1)\lambda - (s-1)(s-5)(-s^2 + 6s + 1) = 0.
$$

With the smallest eigenvalue for coclosed 1-forms being $\lambda = 24$, the indicial roots are

$$
S_3^{\pm} = 3 \pm \sqrt{\pm 3\sqrt{97} + 31}.
$$

(b) The equations for $H_{(2,2)}, H_{(3,1)}$ are

$$d_S * H^{cl}_{(2,2)} + d_H * H^{cc}_{(3,1)} = 0$$
$$d_H H^{cl}_{(2,2)} + d_S H^{cc}_{(3,1)} = 0$$

The indicial equations are

$$(\Delta_S^{Hodge} - (2-s)(4-s))H_{(3,1)} = 0,$$

and for $\lambda = 24$ we have

$$S_5^{\pm} = 3 \pm \sqrt{17}.$$

□

With respect to the volume form on $\mathbb{H}^7 \times \mathbb{S}^4$, there is an inclusion of weighted functions and forms. For $\mathrm{Re}(\lambda) > 3$,

$$x^\lambda \mathcal{C}^\infty(M; {}^e\bigwedge^p T^*M) \subset L^2_e({}^e\bigwedge^p T^*M).$$

And this $\mathrm{Re}(\lambda) = 3$ line is the L^2 cutoff line.

The first thing to notice about the indicial roots results is that those indicial roots appear in pairs symmetric to $\mathrm{Re}(s) = 3$, which is the L^2 line. Most of the indicial roots are bounded away from $\mathrm{Re}(s) = 3$, and as the sperical eigenvalues become larger, they are bounded further. However, there are three pairs of roots that are on the L^2 line, which correponds to the kernel space.

2.3.3 Fredholm property

Now we discuss the behavior of this operator on different eigenspaces, according to whether the pair of indicial roots appear on the L^2 line or off of it. We will use the edge calculus to deal with the large eigenvalues and the 0-calculus to deal with the individual small eigenvalues. Specifically, we will construct two right inverses of this operator to deal with the fact that the kernel appears when the domain becomes larger than L^2.

We will show that for any weight δ off a discrete set of indicial roots, the operator dQ acting on $x^\delta H^{s,k}_{e,b}(M;W)$ is Fredholm. Moreover, it is injective when $\delta > 1$ and surjective when $\delta < -1$. In the range of $\delta \in (-1,0)$, the kernel is finite dimensional, characterized by the three subspaces corresponding to the three indicial roots.

First of all, we define the domain for the linearized operator:

Definition 2.3. *Fix $\delta \in (0,1)$, define the domain as*

$$D_k(\delta) = \{u \in x^{3-\delta} H^{2,k}_{e,b}(M;W) : (dQ + i\epsilon)u \in x^{3+\delta} H^{0,k}_{e,b}(M;W)\}.$$

Using the projection operator π_λ above, the domain can be decomposed in terms of the spherical harmonic decomposition on $\mathbb{S}^4$, as

$$D_k(\delta) = \oplus_{\lambda \in \Lambda} D_k(\lambda, \delta),$$

where Λ is the set of eigenvalues on the 4-sphere. This set is divided into the following three subsets:

- Eigenvalues on functions: $4k(k+3)$;
- Eigenvalues on closed one-forms: $4(k+1)(k+4)$;
- Eigenvalues on coclosed one-forms: $4(k+2)(k+3)$.

We will separately discuss two parts. One part is the infinite dimensional subspace formed by large eigenvalues

$$\oplus_{\lambda > M} D_k(\lambda, \delta),$$

on which the operators $dQ \pm i\epsilon$ are isomorphisms, and approach two limits $D_\pm$ uniformly as ϵ goes to zero. This is shown by using ellipticity and a parametrix construction. The other part is discussed for each small eigenvalue since there are only finitely many. For most of the λ_i's, the operator has the same behavior as the "large" part. The rest correspond to indicial roots lying on the L^2 line, and we use scattering theory to construct resolvents $R_\pm$ and discuss their null spaces.

2.3.4 Large eigenvalues

Consider the bundle $W = \mathrm{Sym}^2(M) \oplus \bigwedge^4(M)$ over $M = \mathbb{B}^7 \times \mathbb{S}^4$ which carries a unitary linear action of SO(5) covering the action on $\mathbb{S}^4$. There is an induced action of SO(5) on $\dot{\mathcal{C}}^\infty(\mathbb{B} \times \mathbb{S}^4; W)$, which extends to all the weighted hybrid Sobolev spaces $x^s H^{k;l}_{\mathrm{e,b}}(\mathbb{B}^7 \times \mathbb{S}^4; W)$ since the group acts through diffeomorphisms. The linearized operator $dQ \in \mathrm{Diff}^2_{\mathrm{e}}(\mathbb{B}^7 \times \mathbb{S}^4; W)$ is an elliptic edge operator for the product edge structure and we have shown that dQ commutes with the induced action of SO(5) on $\dot{\mathcal{C}}^\infty(\mathbb{B}^7 \times \mathbb{S}^4; W)$.

The Sobolev spaces of sections of W decompose according to the irreducible representations of SO(5), all finite dimensional and forming a discrete set. In particular these may be labelled by the eigenvalues, λ, of the Casimir operator for SO(5) with a finite dimensional span when λ is bounded above. The SO(7, 1) action on $\mathbb{H}^7$ commutes with the SO(5) action on W and acts transitively on $\mathbb{H}^7$, so the multiplicity of the SO(5) representation does not vary over $\mathbb{H}^7$. The individual representations of SO(5) in the decomposition of W therefore form bundles over $\mathbb{H}^7$. Therefore we have the following lemma:

Lemma 2.9. *The group* SO(5) *acts on* $x^\delta H^{s,k}_{e,b}(M; W)$ *transitively, and the bundle decomposes to subbundles on* $\mathbb{H}^7$.

And we are going to show the following proposition for projection off finitely many small eigenvalues.

Proposition 2.5. *There is a sifficiently large* $M > 0$*, such that for* $\lambda > M$ *and any* $\epsilon > 0$ *the two operators* $dQ^\lambda \pm i\epsilon$ *acting on* $D^\lambda_k(\epsilon)$ *are both isomorphisms. And their inverses approaches two operators uniformly as* $\epsilon \to 0$.

To prove this proposition, we will bundle all the large eigenvalues together.

Definition 2.4. *For* $\lambda \in [0, \infty)$*, let* $\pi_{\geq\lambda} : W \to W$ *be defined as the the projection off the span of the eigenspaces of the Casimir operator for* SO(5) *with eigenvalues smaller than* λ*, i.e.* $\pi_{\geq\lambda} := \mathrm{Id} - \sum_{\lambda'<\lambda} \pi_{\lambda'}$.

Proposition 2.6. *For any weight $s \in \mathbb{R}$ and any orders k, l, the bounded operator defined as*

$$dQ : x^s H_{e,b}^{k+m,l}(H \times S; W) \to x^s H_{e,b}^{k,l}(H \times S; W)$$

is such that $\pi_{\geq\lambda} dQ$ is an isomorphism onto the range of $\pi_{\geq\lambda}$ for some $\lambda \in [0, \infty)$ (depending on s but not on k and l). Moreover, the range of $\mathrm{Id} - \pi_{\geq\lambda}$ *on $C^\infty(\mathbb{H} \times \mathbb{S}^4; W)$ is the space $C^\infty(M; \oplus_{\lambda'<\lambda} W_{\lambda'})$ of sections of a smooth vector bundle over M and dQ restricts to it as an elliptic element of* $\mathrm{Diff}_0^m(M; \oplus_{\lambda'<\lambda} W_\lambda)$.

To prove this proposition, we first construct an SO(5)-invariant parametrix in the small edge calculus by finding a appropriate kernel on the edge streched product space X_e^2 which is defined from X^2 by blowing up the fiber diagonal.

Definition 2.5. *The edge stretched product X_e^2 for an edge manifold X is defined as the blow up $[X^2; S]$ where S is consists of all fibres of the product fibration $\pi^2 : (\partial X)^2 \to Y^2$ which intersect the diagonal of $(\partial X)^2$.*

Notice that from the definition of fiber diagonal, the blow up actually preserves the product structure of $\mathbb{H}^7 \times \mathbb{S}^4$, i.e. the fibred diagonal contained in X^2 is just the product of $\Delta_e \times S^4 \times S^4$, and the manifold after the blow up is $[(H^7)^2; \partial\Delta] \times (S^4)^2$.

Lemma 2.10. *For $M = \mathbb{H}^7 \times \mathbb{S}^4$, the edge stretched product is actually a product: $X_e^2 = [(\mathbb{H}^7)^2, \partial\Delta] \times (\mathbb{S}^4)^2$.*

The elliptic element dQ is transversely elliptic to the fiber diagonal. Therefore we have a parametrix construction in the small edge calculus as follows.

Lemma 2.11. *Any SO(5)-invariant elliptic operator $dQ \in \mathrm{Diff}_e^m(M; W)$ has an* SO(5)*-invariant parametrix $\tilde{E}$ in $\Psi_e^{-m}(M; W)$, such that*

$$\mathrm{Id} - dQ \circ \tilde{E}, \ \mathrm{Id} - \tilde{E} \circ dQ \in \Psi_e^{-\infty}(M; W)$$

are also SO(5)*-invariant.*

Proof. Any elliptic edge differential operator has a parameterix in the small edge calculus, following Mazzeo [25]. The construction gives the kernel of E as a classical conormal distribution with respect to the 'lifted diagonal' of the stretched edge produce M_e^2. This latter manifold is constructed by blow-up of the fibre diagonal (for the product fibtration $M = \mathbb{H}^7 \times \mathbb{S}^4$) over the boundary of M. In fact, globally in terms of the product this is just the diagonal of hyperbolic space over the boundary, i.e.

$$M_e^2 = (\mathbb{H}^7)_0^2 \times (\mathbb{S}^4)^2$$

where the first space is the zero-stretched product for hyperbolic space. Thus in fact the action of SO(5) on the kernel E, through the product action on M^2, lifts smoothly to M_e^2 and preserves the lifted diagonal (which is the closure of the diagonal in the interior). So we may average under the product action and define

$$\tilde{E} = \int_{g \in SO(5)} g \cdot E.$$

Since dQ is $SO(5)$ invariant by assumption (which is verified for the supergravity operator), $\tilde{E}$ is also a parametrix,

$$dQ \circ \tilde{E} = \operatorname{Id} + \tilde{R},$$

and the average remainder $\tilde{R}$ is also SO(5) invariant. □

As a consequence, now $\tilde{E}$ and $\tilde{R}$ both commute with the spherical eigenvalue projection $\pi_{\geq \lambda}$. The remainder $\tilde{R}$ can be characterized as:

Lemma 2.12. *The Schwartz kernel of $\tilde{R}$ is in $C^\infty((\mathbb{S}^4)^2, \Psi_0^{-\infty}(\mathbb{H}^7) \otimes \operatorname{Hom}(W)) \subset C^\infty(M_e^2, W)$. In consequence it is a smooth map from $(\mathbb{S}^4)^2$ to bounded operators on $x^s H_0^p(H; W)$ for any s, p, with a norm depending on some C^k norm for any bounded range of s.*

Proof. As an element in $\Psi_e^{-\infty}(M; W)$, the Schwartz kernel $\tilde{R}$ is smooth on the double edge space M_e^2, with values in the bundle $\operatorname{Hom}(W) \otimes K$ where K is the kernel density bundle. From the properties of the small calculus, $\tilde{R}$ vanishes to infinite order at the left and right boundary faces. Because M_e^2 has the product structure $(\mathbb{S}^4)^2 \times \mathbb{H}_0^2$, the

Schwartz kernel is in $C^\infty((\mathbb{S}^4)^2, C^\infty(\mathbb{H}_0^2, \mathrm{Hom}(W)\otimes K))$ where $C^\infty(\mathbb{H}_0^2, \mathrm{Hom(W)} \otimes \mathrm{K})$ is the $\Psi_0^{-\infty}(\mathbb{H}; W))$ operators acting on W.

Consider the map from $\Psi_0^{-\infty}(H;W)$ to bounded operators on $x^s H_0^k(\mathbb{H}; W)$: since it is a continuous map from a Fréchet space to a normed space, the norm is bounded by some norm on $\Psi_0^{-\infty}(\mathbb{H}, W)$, i.e. the operator norm of $\tilde{R}$ on $x^s H_0^k(\mathbb{H}; W)$ is bounded by a constant $C(s)\|\tilde{R}\|_{C^k(H;W)}$. For any bounded interval $s \in [-S, S]$, the bound of $\|\tilde{R}\|_{x^s H_0^k(H;W)}$ is uniform. $\square$

We can use the following interpolation result to show that $\pi_\lambda \tilde{R}$ rapidly decays as λ tends to infinity.

Lemma 2.13. $x^s H_{e,b}^{p,k}(M) = L^2(S^4; x^s H_0^p(H)) \cap H^{p+k}(S^4; x^s L_0^2(H))$.

Proof. We only prove the case $s = 0$. If a function f is in $L^2(S, H_0^p(H))$, that is $V_e^p(f) \in L^2(M)$, then applying an elliptic k-th order differential b-operator to f we obtain an element in $H^p(S^4, L^2(H))$, therefore by elliptic regularity $f \in H_{e,b}^{p,k}(M)$. $\square$

Lemma 2.14. *As λ tends to infinity, the bounded operators $\pi_{\geq\lambda}\tilde{R}$ decay in any Sobolev norm $x^s H_{e,b}^{p,l}(M; W)$, i.e.*

$$\lim_{\lambda\to\infty} \|\pi_{\geq\lambda}\tilde{R}\|_{x^s H_{e,b}^{p,l}(H\times S; W)} = 0.$$

Proof. Using Plancherel it follows that the Schwartz kernel of $\pi_{\geq\lambda}\tilde{R}$ rapidly converges to 0 in $C^\infty((S^4)^2, x^s L_0^2(H; W))$ and $L^2((S^4)^2, x^s H_0^p(H; W))$. Then we obtain $\|\pi_{\geq\lambda}\tilde{R}\| \to 0$ as bounded operators on $x^s H_{e,b}^{p,l}(M; W)$ by the above lemma. $\square$

As a consequence, for any fixed s, k, l, there is a λ_0 such that $\|\pi_{\geq\lambda_0}\tilde{R}\|_{x^s H_{e,b}^{k,l}(M;W)} \leq \frac{1}{2}$, and this λ_0 only depends on some C^k norm. In the case that $\pi_{\geq\lambda_0}\tilde{R}$ is small, we get that $\pi_{\geq\lambda_0} dQ \pi_{\geq\lambda_0}\tilde{E}$ is a perturbation of the identity, which is still an isomorphism, that is,

Lemma 2.15. *For any s, k, l, there is a λ_0 depending only on s, such that*

$$\pi_{\geq\lambda_0} dQ \pi_{\geq\lambda_0}\tilde{E} = Id_{\pi_{\geq\lambda_0}W} + \pi_{\geq\lambda_0}\tilde{R}$$

where the right hand side is an isomorphism from $x^s H^{k,l}_{e,b}(M;W)$ to itself.

Proof. The norm of the operator on the right hand side acting on $x^s H^{k,l}_{e,b}(M;W)$ is bounded from 0. □

From the above lemma, we get that $\pi_{\geq\lambda_0} dQ$ is an isomorphism mapping from $\pi_{\geq\lambda_0} x^s H^{k,l}_{e,b}(M;W)$ to $\pi_{\geq\lambda_0} x^s H^{k-m,l}_{e,b}(M;W)$, proving the first part of proposition 2.6.

2.3.5 Individual eigenvalues with $\lambda \neq 0, 16, 40$

Now we consider those eigenvalues smaller than λ_0. Consider the projected operator $\pi_\lambda dQ \pi_\lambda$, which is viewed as a 0-problem on the tensor bundles on $\mathbb{H}^7$ (tensored with fixed eigenforms on $\mathbb{S}^4$). Consider the operator dQ on the space $\pi_\lambda(x^\delta H^{2,k}_{e,b})(M;W)$: computation shows that for a fixed λ, except for $\lambda = 0, 16, 40$, the indicial roots of dQ are contained in the range $(-\infty, \underline{\delta}] \cup [\bar{\delta}, \infty)$ for some $\underline{\delta}, \bar{\delta}$, , so they are separated apart. Moreover the indicial roots are separated further when λ is bigger. With this information, we will show that, For $\lambda > 40$, $dQ^\lambda : \pi_\lambda x^\delta H^{s,k}_{e,b}(M;W) \to \pi_\lambda x^\delta H^{s-2,k}_{e,b}(M;W)$ is Fredholm for any δ.

We consider two operator related to dQ^λ: the normal operator and reduced normal operator.

Definition 2.6 (Mazzeo, [25])**.** *For $L \in \mathrm{Diff}^*_e(X)$ the normal operator $N(L)$is defined to be the restriction to the front face B_{11} of the lift of L to X^2_e. In terms of the local coordinate, if*

$$L = \sum_{j+|\alpha|+|\beta|\leq m} a_{j,\alpha,\beta}(x,y,z)(x\partial_x)^j (x\partial y)^\alpha \partial_z^\beta$$

then

$$N(L) = \sum_{j+|\alpha|+|\beta|\leq m} a(0,\tilde{y},z)(s\partial_s)^j (s\partial_u)^\alpha \partial_z^\beta,$$

where $s, u, \tilde{x}, \tilde{y}, z, \tilde{z}$ is the lifted coordinate system on X^2_e.

Here $N(L)$ acts on the product of $\mathbb{R}^{k+1}_+ \times F$ and is invariant under the linear translations and dilations on the first factor. This may be further reduced to be the reduced normal operator which is a family of differential b-operators.

Definition 2.7. *The reduced normal operator $N_0(L)$ is defined by applying the Fourier transform in the $\mathbb{R}^k$ direction to $N(L)$ then doing a rescaling. Specifically,*

$$N_0(L) = \sum_{j+|\alpha|+|\beta|\leq m} a_{j,\alpha,\beta}(t\partial_t)^j(it\hat{\eta})^\alpha\partial_z^\beta, \quad t \in \mathbb{R}_+, \quad \hat{\eta} \in S^*_{\tilde{y}}Y.$$

Note that in our case, we are interested in the reduced operator of dQ^λ which is independent of the spherical variables z, and therefore is an ordinary differential operator.

Lemma 2.16. *For $\lambda > 40$, there are $\underline{\delta} < 0 < \bar{\delta}$, such that the reduced normal operator $N_0(dQ^\lambda)$ is an isomorphism on $x^\delta L^2(\mathbb{R}^+_s)$ for any $\underline{\delta} < \delta < \bar{\delta}$.*

Proof. $N_0(dQ^\lambda)$ has a pair of indicial roots for each λ, which tend to $\pm\infty$ as λ goes to infinity. For $\lambda \neq 0, 16, 40$, the pair of indicial roots have different real parts and thus can be separated. An ODE operator is not injective on $x^\delta L^2(\mathbb{R}^+_s)$when δ is less than the bottom indicial root and not surjective when δ is bigger than the top indicial root. Since there is a gap between the pairs of indicial roots for $\lambda > 40$, we can find $\underline{\delta}, \bar{\delta}$ such that $N_0(dQ^\lambda)$ is an isomorphism on $x^\delta L^2(\mathbb{R}^+_s)$ for $\underline{\delta} < \delta < \bar{\delta}$. □

Lemma 2.17. *For $\lambda > 40$ and $\underline{\delta} < \delta < \bar{\delta}$, the normal operator $N(dQ^\lambda)$ is Fredholm on $x^\delta H^k_b(\mathbb{H}^7; \pi_\lambda W)$.*

Proof. The reduced normal operator is obtained by Fourier transform and normalization of the operator $N(P_\lambda)$, so we may do an inverse Fourier transform to get back to N from N_0.

From Mazzeo [25], there exists a parametrix G and two projectors P_i, such that the Schwartz kernels $k(G)$ and $k(P_i)$ all lift to distributions on X^2_e polyhomogeneous conormal at all boundary faces. The exponents in the expansions are determined by the indicial roots. □

Remark 1. *On M we have the following inclusion:*

$$x^{3+\delta}C^\infty(M; W) \subset x^\delta L^2_e(M; W).$$

The reason why the line of indicial roots with $\mathrm{Re}(s) = 3$ *is important is that there is a symmetry with respect to this line which is related to the self-adjointness of the operator.*

Lemma 2.18. *The kernel of the normal operator* $N(dQ^\lambda)$ *on* $x^\delta H_b^k(\mathbb{H}^7; \pi_\lambda W)$ *is zero for* $\delta > 0$.

Proof. This follows from the fact that there are no finite dimensional L^2 eigenspaces for functions and tensors on $\mathbb{H}^7$.Indeed, consider the representation of $SO(7,1)$ on tensor bundles on $\mathbb{H}^7$. There are no finite dimensional L^2 invariant subspace of forms on $\mathbb{H}^7$. For tensors, from Delay's result [7], there are no L^2 eigentensors. □

As a result we have the following:

Lemma 2.19. *For any* $\delta > 0$, *the normal operator* $N(dQ^\lambda)$ *is injective on* $x^\delta H_0^k(M; \pi_\lambda W)$ *and surjective on* $x^{-\delta} H_0^k(M; \pi_\lambda W)$.

Proof. The kernel of this map on $x^\delta H_0^k(M; \pi_\lambda W)$ is contained in L^2 eigenspace of forms and tensors on $\mathbb{H}^7$, which from the lemma above does not have any nontrivial elements. Therefore it is injective. By duality, it's surjective on the bigger space $x^{-\delta} H_0^k(M; \pi_\lambda W)$. □

We now return to the original operator dQ^λ and show that it is Fredholm.

Proposition 2.7. *For any* $\delta > 0$, *the operator* $dQ^\lambda : x^{3+\delta} H_{e,b}^{2,k}(M; \pi_\lambda W) \to x^{3+\delta} H_{e,b}^{0,k}(M; \pi_\lambda W)$ *is injective. Likewise, it is surjective on* $x^{3-\delta} H_{e,b}^{2,k}(M; \pi_\lambda W)$.

Proof. The normal operator is an isomorphism at each point of the boundary $\mathbb{S}^6$. Thus for a general kernel element of dQ^λ, we decompose it using $SO(7,1)$ action, so it falls into the kernel space of the normal operator $N(dQ^\lambda)$ for which there is not any. Therefore the kernel is also trivial for the operator dQ^λ. So it is injective on the smaller space, and by duality surjective on the bigger space. □

2.3.6 Individual eigenvalues with $\lambda = 0, 16, 40$

For those eigenvalues corresponding to indicial roots with real part 3, we consider each subspace $\pi_\lambda x^{3-\delta} H^{2;k}_{e,b}(M; W)$ separately. Restricted to these subspaces, the linearized operator is a 0-operator on hyperbolic space, of which the main part is the hyperbolic laplacian $\Delta_{\mathbb{H}}$. From Guillarmou [18], the resolvent of $\Delta_{\mathbb{H}} - \lambda$, denoted as $R(\lambda)$, extends to a meromorphic family with finite degree poles. Similarly, we want to show that dQ has two generalized inverses $R_\pm$, which is the extension of the resolvent $(dQ \pm i\epsilon)^{-1}$ when ϵ approaches the real axis. More specifically, we will prove the following result that, for $\lambda = 0, 16, 40$, $dQ^\lambda : x^{-\delta} H^s_0(\mathbb{H}^7; \pi_\lambda W) \to x^\delta H^{s-2}_0(\mathbb{H}^7; \pi_\lambda W)$ is bounded and has two generalized inverses.

We will be using indicial roots analysis again here, but first we will need to show that the indicial roots may be separated from the L^2 line by perturbing the operator.

Lemma 2.20. *For $\lambda = 0, 16, 40$, the two indicial roots of operator $dQ^\lambda \pm i\epsilon$ lie off the* $\mathrm{Re}(s) = 3$ *line.*

Proof. Suppose $s \in \mathbb{C}$ is an indicial root for an operator P on a point p at the boundary, then we have $P(x^s) = O(x^{s+1})$ by definition. For $\epsilon \neq 0$, the following computation shows that s is no longer an indicial root: $(P + i\epsilon)(x^s) = i\epsilon x^s + O(x^{s+1}) \notin O(x^{s+1})$. Instead, take the harmonic 4-form part which has inidicial roots $3 \pm 6i$ which in the indicial root computation is $P(x^{3+s}) = (s^2 + 36)x^{3+s} + O(x^4)$, after perturbation it becomes

$$(P + i\epsilon)(x^{3+s}) = (s^2 + i\epsilon + 36)x^{3+s} + O(x^{s+1})$$

so the indicial equation becomes $s^2 = -i\epsilon - 36$ which moves the two roots $3 + s_\pm$ off the line of $Re(s) = 3$. A similar argument applies to other two pairs of roots. □

Lemma 2.21. *For $\epsilon \neq 0$, the inverse $(dQ^\lambda \pm i\epsilon)^{-1} : x^{-\delta} H^{0,k}_{e,b}(M; \pi_\lambda W) \to x^\delta H^{2,k}_{e,b}(M; \pi_\lambda W)$ exists as a bounded operator.*

Proof. Using the indicial roots separation and same argument as before for those eigenvalues greater than 40, the operator $dQ^\lambda \pm i\epsilon$ is Fredholm on $x^\delta H^{2,k}_{e,b}(M; \pi_\lambda W)$, injective on the smaller space and surjective on the larger space. □

We need the following limiting absorption principle:

Proposition 2.8. *For small weight* $0 < \delta < 1$*, and number* s, k*, the operators* $(dQ^\lambda \pm i\epsilon)^{-1}$ *converges uniformly to bounded operators on weighted space,*

$$\lim_{\epsilon \to 0} \|(dQ_\lambda \pm i\epsilon)^{-1} - R_\pm^\lambda\| = 0.$$

where $R_\pm^\lambda : x^\delta H_{e,b}^{s,k}(M; \pi_\lambda W) \to \pi_\lambda x^\delta H_{e,b}^{s-2,k}(M; \pi_\lambda W)$*.*

Proof. To prove this, we will consider the reduced normal operator of $dQ_\lambda - i\epsilon$, which is a differential operator (parametrized by y and ϵ), is injective from $x^\delta H^2(\mathbb{R}^+) \to x^\delta L^2(\mathbb{R}^+)$ for any fixed δ. This ODE operator may be extended holomorphically as ϵ passes through zero from above, and the solution of the ODE extends holomorphically as well. After extending it past zero, the smaller indicial root moves into the larger one. $dQ_\lambda \pm i\epsilon$ is injective on $x^{3+\delta} H_{e,b}^{s,k}(M, W) \to x^{3+\delta} H_{e,b}^{s-2,k}(M, W)$ (as it excludes half the roots), and the resolvent $R_+ := \lim_{\epsilon \to 0}(dQ_\lambda - i\epsilon)^{-1}$ is an right inverse. Similarly for R_-. □

2.3.7 Boundary data for the linear operator

Combining the analysis for λ off the L^2 line and on the L^2 line, we conclude the following for dQ:

Proposition 2.9. *For* $\delta \in (0, 1)$*, there are two generalized inverses* $R_\pm : x^\delta H_{e,b}^{s,k}(M; W) \to x^{-\delta} H_{e,b}^{s+2,k}(M; W)$ *for operator* dQ*, such that*

$$P \circ R_+ = Id, P \circ R_- = Id : x^\delta H_{e,b}^{s,k}(M; W) \to x^\delta H_{e,b}^{s,k}(M; W).$$

As a consequence, we find the following right inverse which is real:

$$(dQ)^{-1} := \frac{1}{2}(R_+ + R_-).$$

To get the main theorem, we will parametrize the domain by the boundary data below to get a family of operators Q_v. To show Q_v is a local isomorphism, we use an implicit function theorem argument that, for small boundary data v, $Q_v \circ (dQ_0)^{-1}$, which acts on a fixed space independent of v, is a perturbation of the identity.

First we define three bundles on $\mathbb{S}^6$ that parametrize the incoming and outgoing boundary data for the linear operator.

Definition 2.8. *Let $V_1^{\pm}$ to be the space of sections of the bundle of 3-forms with $*_S$ eigenvalue $\pm i$:*

$$V_1 := \{v_1 \in C^\infty(\mathbb{S}^6; \bigwedge^3 T^*\mathbb{S}^6) : *_{\mathbb{S}^6} v_1 = i v_1\}.$$

Similarly let $V_2^{\pm}$ and $V_3^{\pm}$ be the smooth functions on the 6-sphere tensored with eigenforms on 4-sphere:

$$V_2 := \{v_2 \otimes \xi_{16} : v_2 \in C^\infty(\mathbb{S}^6; \mathbb{R}), \xi_{16} \in E^{cl}_{16}(\mathbb{S}^4)\},$$

$$V_3 := \{v_3 \otimes \xi_{40} : v_3 \in C^\infty(\mathbb{S}^6; \mathbb{R}), \xi_{40} \in E^{cl}_{40}(\mathbb{S}^4)\}.$$

Remark 2. *Note the dimension of the closed 1-form with the first and second eigenvalues are determined by the degree 2 and 3 spherical harmonics with 4 variables, which, repectively, are 5 and 14 dimensional vector spaces.*

To save some space we will use the following abbreviation for the leading expansion given by the three parameters.

Definition 2.9 (Leading expansion for the linear operator)**.** *When we say the leading expansion is given by $\sum_{i=1}^3 v_i^+ \xi_i$, we will mean*

$$\begin{aligned}
H_{(4,0)} &= v_1^+ \xi_1 x^{\theta_1^+} + S_1(v_1^+)\xi_1 x^{\theta_1^-} + O(x^{3+\epsilon}) \\
\operatorname{Tr}_{\mathbb{H}^7} g &= \operatorname{Tr}_{\mathbb{H}^7} h + 7 *_s (v_2^+ \xi_2 x^{\theta_2^+} + S_2(v_2^+)\xi_2 x^{\theta_2^-} + v_3^+ \xi_3 x^{\theta_3^+} + S_3(v_3^+)\xi_3 x^{\theta_3^-}) + O(x^{3+\epsilon}) \\
\operatorname{Tr}_{\mathbb{S}^4} g &= \operatorname{Tr}_{\mathbb{S}^4} h + 4 *_s (v_2^+ \xi_2 x^{\theta_2^+} + S_2(v_2^+)\xi_2 x^{\theta_2^-} + v_3^+ \xi_3 x^{\theta_3^+} + S_3(v_3^+)\xi_3 x^{\theta_3^-}) + O(x^{3+\epsilon}) \\
g_{(1,1)} &= h_{1,1} + (v_2^+ \xi_2 x^{\theta_2^+} + S_2(v_2^+)\xi_2 x^{\theta_2^-} + v_3^+ \xi_3 x^{\theta_3^+} + S_3(v_3^+)\xi_3 x^{\theta_3^-}) + O(x^{3+\epsilon}) \\
H_{(1,3)} &= -d_H (v_2^+ \xi_2 x^{\theta_2^+} + S_2(v_2^+)\xi_2 x^{\theta_2^-} + v_3^+ \xi_3 x^{\theta_3^+} + S_3(v_3^+)\xi_3 x^{\theta_3^-}) + O(x^{3+\epsilon}) \\
H_{(0,4)} &= 6 \operatorname{Vol}_{\mathbb{S}^4} + d_s *_s (v_2^+ \xi_2 x^{\theta_2^+} + S_2(v_2^+)\xi_2 x^{\theta_2^-} + v_3^+ \xi_3 x^{\theta_3^+} + S_3(v_3^+)\xi_3 x^{\theta_3^-}) + O(x^{3+\epsilon})
\end{aligned} \tag{2.45}$$

Theorem 2.3. *For any Poincaré–Einstein metric h that is close to the background metric g_0, the solution to the linearized equations $dQ(g,H) = 0$ is parametrized by the data of the three bundles $\oplus V_i$.*

To prove the theorem above, we will first work on hyperbolic space, and use a perturbation argument to show that, for a nearby Poincaré–Einstein metric, the parameter space is also transversal to the kernel. For the weight $3-\delta$ where $\delta > 0$ and is small, the operator dQ is surjective but not injective, and there is a null space which corresponds to the indicial roots with real part 3.

The scattering matrix relates the incoming and outgoing data of eigenfunctions corresponding to a point in the continuous spectrum. Once we fix the incoming data in the expansion, the outgoing data is determined by the scattering matrix. In the hyperbolic metric case, the scattering matrix $S_i(s), i = 1,2,3$ is defined for each pair of special indicial roots.

The scattering matrix in the hyperbolic case is

$$S_i(s) : C^\infty(\partial\mathbb{H}^7; V_i^+) \to C^\infty(\partial\mathbb{H}^7; V_i^-) \tag{2.46}$$

with property that if

$$dQ(u) = 0, u = \sum_{i=1}^{3} f_i x^{\theta_i^+} + g_i x^{\theta_i^-} + O(x^{3+\epsilon}),$$

then

$$g_i|_{\partial M} = S_i(s) f_i.$$

Proposition 2.10. *For the base case with hyperbolic metric, the kernel of operator dQ is parametrized by the sections of $\oplus V_i$. More specifically, for any small incoming real data $v^+ = (v_1^+, v_2^+, v_3^+)$, there is a unique solution to the linearized equations with $\sum_{i=1}^{3} \xi_i (v_i^+ x^{\theta_i^+} + S_i(v_i^+) x^{\theta_i^-})$ as the leading expansion.*

Proof. We use the result of Graham–Zworski [16] and Guillarmou–Naud [19] about the description of scattering matrix in hyperbolic space:

$$S(s) = 2^{n-2s}\frac{\Gamma(\frac{n}{2}-s)}{\Gamma(s-\frac{n}{2})}\frac{\Gamma\left(\sqrt{\Delta_{\mathbb{S}^n}+(\frac{n-1}{2})^2}+\frac{1-n}{2}+s\right)}{\Gamma\left(\sqrt{\Delta_{\mathbb{S}^n}+(\frac{n-1}{2})^2}+\frac{n+1}{2}-s\right)},$$

where if we put in $s=\theta_1^+=3+6i$ we get

$$S(3+6i) = 2^{-12i}\frac{\Gamma(-6i)}{\Gamma(6i)}\frac{\Gamma(\sqrt{\varDelta_{\mathbb{S}^6}+\frac{25}{4}}+\frac{1}{2}+6i)}{\Gamma(\sqrt{\varDelta_{\mathbb{S}^6}+\frac{25}{4}}+\frac{1}{2}-6i)}.$$

Since the scattering matrix is a function of the laplacian on the boundary S^6 we can take the eigenvalue expansion on 6-sphere with real eigenform f_λ, we would consider the following expression, which is real and forms the leading order of the actual solution:

$$x^{3+6i}f_\lambda + x^{3-6i}S(3+6i)f_\lambda = x^{3+6i}f_\lambda + x^{3-6i}(2^{-12i}e^{i2\theta}\lambda^{6i})f_\lambda.$$

Here θ is a real number determined by

$$e^{2i\theta(\lambda)} = \frac{\Gamma(-6i)}{\Gamma(6i)}\frac{\Gamma\left(\sqrt{\lambda+\frac{25}{4}}+\frac{1}{2}+6i\right)}{\Gamma\left(\sqrt{\lambda+\frac{25}{4}}+\frac{1}{2}-6i\right)}, \tag{2.47}$$

by using the relation of

$$\Gamma(\bar{z}) = \overline{\Gamma(z)}$$

so that the right hand side of (2.47) is a complex number with norm 1 and θ is a real number determined by λ.

Rearranging the expression, the solution in the eigenvalue λ component is

$$\pi_\lambda u = x^{3+6i}f_\lambda + x^{3-6i}2^{-12i}e^{2i\theta}f_\lambda \tag{2.48}$$

$$= x^3 2^{-6i}e^{i\theta}\left((2x)^{6i}e^{i\theta}+(2x)^{-6i}e^{i\theta}\right)f_\lambda \tag{2.49}$$

$$= x^3 2^{1-6i}e^{i\theta}\,\mathrm{Re}\left((2x)^{6i}e^{i\theta(\lambda)}\right)f_\lambda \tag{2.50}$$

which is a product of a real 3-form with complex constant $x^3 2^{1-6i} e^{i\theta}$. Therefore in this case,

$$f = |f| e^{it}, t \in \mathbb{R},$$

and f is determined by a real 3-form.

A similar computation shows that $\theta_2^\pm, \theta_3^\pm$ are parametrized by real functions. □

With the computation above, we have shown that the leading expansion of the solution has the form

$$u = v_1^\pm \wedge dx/x \xi_1 x^{\theta_1^\pm} + \xi_2 v_2^\pm x^{\theta_2^\pm} + \xi_3 v_3^\pm x^{\theta_3^\pm} + O(x^{3+\delta})$$

where

$$v_i^\pm \in V_i^\pm, v_i^- = S_i v_i^+.$$

Definition 2.10. *We define the Poisson operator P for a Poincaré–Einstein metric h: if we denote the operator that maps from the space of real solutions to the incoming boundary data (v_1^+, v_2^+, v_3^+) by f, then the inverse of this map is denoted by P, which is the Poisson operator:*

$$P = f^{-1} : \oplus V_i \to D_{v,h}, \quad \{v_i\} \mapsto \sum_{i,s} \mathrm{Re}(v_i^+ x^{3+is} + S_i(s) v_i^+ x^{3-is}) \in x^{3-\delta} H_b^\infty(M; W).$$

For the hyperbolic case, it maps to the actual solution with leading expansion (v_i). For nearby Poincaré–Einstein metrics, it maps to a real element in the domain which is not necesssarily an element in the null space, however it is very close to a null element with the same leading expansion. We show below that it is transversal to the range of right inverse $R_+ + R_-$.

We also remark here that, the domain and range of this operator P are real, and for nearby Poincaré–Einstein metrics, the composition $Q \circ P(u)$ also maps into real space.

Proposition 2.11. *For the base case, the real null space of dQ is the range of $i(R_+ - R_-)$.*

Proof. This is Stone's theorem. For any element u, $dQ \circ (R_+ - R_-)u = 0$, and any element in the kernel of dQ is parametrized by u. We multiply i to make it real since $R_+ - R_-$ is completely imaginary. □

Lemma 2.22. *For a Poincaré–Einstein metric h that is closed to the background metric g_0, the range space of the sum of two generalized inverses $R_\pm$ is transversal to the range of their difference:* $\mathrm{Range}(R_+ + R_-)$ *is transversal to* $\mathrm{Range}(R_+ - R_-)$.

Proof. For the base case: the range of P is the kernel of dQ, which is also range of $R_+ - R_-$. However, the range of $R_+ + R_-$ doesn't contain any element of the kernel, since $dQ \circ R_+, \quad dQ \circ R_- \neq 0$.

Since transversality is stable under small perturbations, the result follows. □

Then with the two lemmas above, we conclude:

Lemma 2.23. *The range space of the Poisson operator P is transversal to* $\mathrm{Range}(R_+ + R_-)$.

2.4 Nonlinear equations: application of the implicit function theorem

From the discussion of the linear operator dQ above, we now can apply the implicit function theorem to get results for the nonlinear operator. The nonlinear terms include two parts, one from the linearization of the curvature operator, the other from the product type terms. We will use a perturbation argument to show that for each Poincaré–Einstein metric, the nearby solutions are parametrized by the three parameters on $\mathbb{S}^6$ as in the linear case.

To deal with the fact that the domain changes with the base metric and the parameters we put in, we will use an implicit function theorem, that is, constructing a map from range space to itself, and show that this map is a perturbation of identity, therefore an isomorphism.

2.4.1 Domain defined with parametrization

First of all we define the domain for all the product type metrics of a nearby Poincaré–Einstein metric h and each parameter set $v = (v_1, v_2, v_3) \in \oplus_{i=1}^3 V_i$. From the discussion of the generalized inverses of the linearized operator, we know that the image of $\frac{1}{2}(R_+ + R_-)$ is transversal to the image of the Poisson operator which is close to the kernel of the linearized operator for a nearby Poincaré–Einstein metric. In the nonlinear case, we define the domain so that, for each parameter v, it is an affine section translated by Pv, where P is the Poisson operator defined above. The domain has the property that, in the linearized case with base hyperbolic metric, it is mapped by dQ isomorphically back to the range space $x^\delta H^{0,k}_{e,b}(M;W)$.

Definition 2.11. *(Domain of nonlinear operator) For a Poincaré–Einstein metric h that is close to the base hyperbolic metric and a set of parameters $v = (v_1, v_2, v_3)$ in bundle V, the domain $D_{h,v}$ of the nonlinear operator is defined as*

$$D_{h,v} := \left\{ \frac{1}{2}(R_+ + R_-)f + Pv \,\middle|\, f \in x^\delta H^{0,k}_{e,b}(M;W) \right\}.$$

Note that the domain depends on the choice of h and v, where the dependence of h comes from the construction $\frac{1}{2}(R_+ + R_-) = (dQ_h)^{-1}$. One important property of this domain is that $D_{h,v}$ is mapped surjectively to the range space $x^\delta H^{0,k}_{e,b}(M;W)$ by the linear operator dQ_h.

Lemma 2.24. *The range space of the linear map dQ_h acting on $D_{h,v}$ is $x^\delta H^{0,k}_{e,b}$.*

Proof. Since all the operations are linear,

$$dQ_h(\frac{1}{2}(R_+ + R_-)f + Pv) = dQ_h(dQ_h)^{-1}f + dQ_h(Pv) = f + O(x^4).$$

Here we used the fact that R_+ and R_- are both generalized inverses for dQ_h, and the Poisson operator maps into a space with extra decay in x. Since f can be any element in $x^\delta H^{0,k}_{e,b}(M;W)$, it follows that dQ_h maps the domain $D_{h,v}$ onto the range. □

Now with the domain we can define a nonlinear operator $Q_{h,v}$ which is parametrized by the background Poincaré–Einstein metric h and the set of parameters v, which can be viewed as a translation of the original operator $Q_{h,0}$.

Definition 2.12. *We define the parametrized nonlinear operator $Q_{h,v}$ on domain $D_{h,v}$:*

$$Q_{h,v}u := Q_{h,0}(u + Pv)$$

As a translation of the original operator, the linearization of $Q_{h,v}$ at the point $(0,0)$ is the same for the original nonlinear supergravity operator.

Lemma 2.25. *At any boundary point on $\mathbb{S}^6$, the linearization of the parametrized nonlinear operator $Q_{h,v}$ at point $u = (k, H) = (0, 0)$ equals to $dQ_{0,0}$.*

Proof. Since $Q_{h,v}$ is defined as a translation of $Q_{g_0,0}$ by Pv, then near the boundary $\mathbb{S}^6$,

$$dQ_{h,v}(0,0)u = d(Q_{h,0}(u + Pv)) = dQ_{h,0}u + dQ \circ Pv = dQ_{g_0,0}u + O(x)$$

Therefore on any boundary point, we have the same linearization as $dQ_{0,0}$. □

Next we show that the nonlinear terms are well controlled, i.e. mapped into the smaller space $x^\delta H^{2,k}_{e,b}(M; W)$.

Lemma 2.26. *For k sufficiently large, the product type nonlinear terms: $F \circ F - d(F \circ F)$, and $F \bigwedge F - d(F \wedge F)$ are both contained in $x^\delta H^{2,k}_{e,b}(M; {}^e \wedge^4 (M))$.*

Proof. The nonlinear parts are $F \wedge F$ and $F \circ F$ which are products of two elements in the range space $x^\delta H^{2,k}_{e,b}(M, S^2({}^eT^*M)) \oplus x^\delta H^{1,k}_{e,b}(M, \bigwedge^4_{cl}({}^eT^*M))$. With respect to basis of the edge bundles, these may be considered locally as functions in $x^\delta H^{2,k}_{e,b}(M)$. Using the algebra property included in the appendix, we know that for $r > -3$, and s, k, and any $f, g \in x^r H^{s,k}_{e,b}(M)$, the product fg is also in $x^r H^{s,k}_{e,b}(M)$. Since in our case $\delta > 0$, the result follows. □

The last nonlinear term is the remainder from the linearization of Ric, for which we show below that it is also contained in the range space.

Lemma 2.27. *The nonlinear remainder of* Ric, Ric $-d(\mathrm{Ric})$ *is contained in* $x^{3+\delta}H^{0,k}_{e,b}$ $(M;\mathrm{Sym}^2(T^*M))$.

Proof. We compute the linearization $d(\mathrm{Ric})$, which acting on a 2-tensor h can be written as

$$d(Ric)[h] = \frac{-1}{2}g^{ml}(\nabla_m\nabla_l h_{jk} - \nabla_m\nabla_k h_{jl} - \nabla_l\nabla_j h_{mk} - \nabla_j\nabla_k h_{ml}).$$

Comparing Ric and $d(\mathrm{Ric})$, the difference is a 3rd order polynomial of g, g^{-1} and first order derivatives of these with smooth coefficients. Since the metric component g and g^{-1} are smooth, hence in $x^0H^{s,k}_{e,b}(M)$, it follows again by the algebra property that their product is contained in $x^\delta H^{s,k}_{e,b}(M;\mathrm{Sym}^2(M))$. □

The composed operator $Q_{h,v}\circ(dQ_{0,0})^{-1}$ is this well-defined operator as a map on the following space:

$$Q_{h,v}\circ(dQ_{0,0})^{-1} : x^\delta H^{0,k}_{e,b}(M;W) \to x^\delta H^{0,k}_{e,b}(M;W).$$

$$f \mapsto Q_{h,0}\Big(\frac{1}{2}(R_+ + R_-)f + Pv\Big)$$

We now discuss the properties of this operator using the implicit function theorem.

Lemma 2.28 (Implicit function theorem)**.** *Consider the following smooth map* $f : V\times M \to M$ *near a point* $(v_0, m_0) \in V\times M$ *with* $f(v_0, m_0) = c$*, if the linearization of the map with respect to the second variable* $df_2(v_0, m_0) : M \to M$ *is an isomorphism, then there is neighborhood* $v_0 \in U \subset V$ *and a smooth map* $g : V \to M$*, such that* $f(v, g(v)) = c, \forall v \in U$.

Theorem 2.4. *For any* $s \geq 2, k \gg 0$ *there exists* $\delta > 0, \rho > 0$*, such that, for a Poincaré–Einstein metric* h *that is sufficiently close to the base metric* g_0*, for each small boundary value perturbation* $v = \oplus_{i=1}^3 v_i^+$ *with* $\|v\|_{H^k_b(M;\oplus V_i)} < \rho$*, there is a unique solution* $u = (g, H) \in D_{v,h} \subset x^{-\delta}H^{s,k}_{e,b}(M;W)$ *satisfying the supergravity equations*

$Q(u) = 0$ *with the following leading expansion*

$$(g, H) = (h, 6\,\mathrm{Vol}_{\mathbb{S}^4}) + \sum_{i=1}^{3} v_i^+ \xi_i x^{\theta_i} \tag{2.51}$$

To prove the theorem, we will apply the implicit function theorem to the following operator:

$$Q_{h,v} \cdot \circ (dQ_{0,0})^{-1} : \oplus_{i=1}^{3} V_i \times x^{3+\delta} H_{e,b}^{0,k}(M; W) \to x^{3+\delta} H_{e,b}^{0,k}(M; W)$$

$$(v, f) \mapsto Q_{h,v} \circ (dQ_{0,0})^{-1}(f)$$

This map is from a neghborhood of the Banach space $H^k(\mathbb{S}^6, \oplus V_i) \times x^\delta H_{e,b}^{0,k}(M; W)$ to the Banach space $x^{3+\delta} H_{e,b}^{0,k}(M; W)$. The following is a consequence of Lemma 2.25.

Lemma 2.29. *The linearization of* $Q_{h,v} \circ (dQ_{0,0})^{-1}$ *at point* $(v, f) = (0, 0) \in \oplus V_i \times x^\delta H_{e,b}^{0,k}(M; W)$ *is an isomorphism.*

Proof. From Lemma 2.25 we know that at the point $(v, f) = (0, 0) \in \oplus V_i \times x^\delta H_{e,b}^{0,k}(M; W)$ the linearization, which is the composition of linearization of the operators, is

$$d(Q_{h,v} \circ dQ_{0,0})^{-1})_{(0,0)} = \mathrm{id} : x^\delta H_{e,b}^{0,k}(M; W) \to x^\delta H_{e,b}^{0,k}(M; W). \quad \square$$

Lemma 2.30. *For a given metric* h*, the map* $Q_{h,v} \circ (dQ_{0,0})^{-1}$ *as an edge operator varies smoothly with the parameter* $v \in V$*.*

Proof. From the construction of $dQ_{0,0}^{-1}$ we know it is an edge operator. And from the discussion for $Q_{h,0}$, this nonlinear operator is also edge. Now we we only need to show that when the nonlinear operator Q applies to elements of type $f + Pv$, it varies smoothly with the parameter v. This follows from the algebra property and the fact that a second order elliptic edge operator maps from $H_e^s(M)$ to $H_e^{s-2}(M)$ smoothly as shown in the appendix. $\square$

We now obtain the following, as a direct result of the implicit function theorem.

Proposition 2.12. *There are $\rho_1, \rho_2 > 0$, such that on the two neighborhoods $U_1 := \{v \in \oplus V_i | \|v_i\|_{H^k} < \rho_1\}$ and $U_2 := \{f \in x^{3+\delta} H^{0,k}_{e,b}(M;W) | \|f\|_{x^{3+\delta} H^{0,k}_{e,b}(M;W)} < \rho_2\}$, there exists a continuous differentiable map $g : U_1 \to U_2$ such that*

$$Q_{h,v} \cdot (dQ_{0,0})^{-1}(g(v)) = 0.$$

Proof. Now we use the implicit function theorem, we can find neighborhoods of $v = 0$ and $f = 0$, in this case, U_1 and U_2 such that the nonlinear map $Q_{h,v} \circ (dQ_{0,0})^{-1}$ is a bijective smooth map on U_2 for any $v \in U_1$. And this gives us the parametrized map g from U_1 to U_2. □

With the proposition above, we find a solution for each set of parameter $\{v_i\}$.

Proof of Theorem 2.4 . Using the definition of $Q_{h,v} \circ (dQ_{0,0})^{-1}(g(v)) = 0$ with the map g constructed above, we can rewrite it as

$$Q_h(dQ^{-1}(g(v)) + Pv) = 0.$$

That is, for each parameter set v, $u = dQ^{-1}(g(v)) + Pv$ is the unique solution in the space $D_{v,h} \subset x^{-\delta} H^{s,k}_{e,b}(M;W)$. □

2.4.2 Regularity of the solution

Next we show that the solution obtained above is smooth if the boundary data is smooth.

Proposition 2.13. *If the boundary data $v \in C^\infty(\mathbb{S}^6, \oplus V_i)$, then the solution u is in $H^\infty_b(M;W)$.*

Proof. This is done by elliptic regularity. We would like to prove that for any k,

$$\|u\|_{H^{k+2}_b(M;W)} \leq C(\|v\|_{H^k(S^6;V_i)} + \|Q(u)\|_{H^{0,k}_{e,b}(M;W)}).$$

Since the principal part of Q is the elliptic edge operator $\oplus_\lambda dQ_\lambda$ and for each λ, we have such elliptic estimates

$$\|u_\lambda\|_{H_b^{k+2}(M;W)} \leq C(\|v_\lambda\|_{H^k(S^6;V_i)} + \|dQ_\lambda(u)\|_{H_{e,b}^{0,k}(M;W)}).$$

The nonlinear parts are lower order:

$$\|Q - \sum DQ_\lambda\|_{H_b^{k+2}(M;W)} < C,$$

leading to an elliptic estimate for Q. □

We can also obtain a classical expansion of the solution. The leading terms are given by the combination of incoming and outgoing boundary data from V_i, and lower order terms are solved by iteration.

Proposition 2.14. *The solution has a classical polyhomogeneous expansion, with leading term*

$$u = \sum_{i=1}^{3} v_i^{\pm} x_i x^{3+\theta_i^{\pm}} + \sum_{j\geq 4}\sum v_j x^j \Big(\sum_{k\leq j} (\log x)^k f_k\Big)$$

where v_j are eigenforms, and $f_k \in \mathcal{C}^\infty(M, W)$. For the lower order terms, the exponent of the logarithmic terms grows linearly with the order.

Proof. We solve the problem iteratively. For the first order problem, from the linearization and its inverse construction, we have $u_1 = \sum v_i^{\pm} x^{3\pm\theta_i} x_{1,i}$ with

$$Q(u_1) = x^{3+\delta} e_1, \quad e_1 \in \mathcal{C}^\infty(M, W).$$

Then we solve away the $x^\delta e_1$ term and lower indicial roots appear here, which gives us

$$u_2 = x^3(\sum_i v_i^{\pm} x^{3\pm\theta_i} x_{i,2} + x \log x).$$

The log terms appear because an order in the expansion of $x^\delta e$ coincides with one of the indicial roots. Then iteratively we obtain the terms

$$u_j = \sum_{j\geq 4}\sum v_j x^j (\sum_{k\leq j} (\log x)^k f_k),$$

where each time the power of log increases by at most one. □

In terms of the explicit formulae, we may summarize the previous results as follows:

Theorem 2.5. *The solution as we get from given boundary data $v_i^{\pm}$ is polyhomegeneous and has the following expansion:*

$$\begin{aligned}
H_{(4,0)} &= v_1^+ \xi_1 x^{\theta_1^+} + S_1(v_1^+)\xi_1 x^{\theta_1^-} + O(x^{3+\epsilon}) \\
\mathrm{Tr}_{\mathbb{H}^7}\, g &= \mathrm{Tr}_{\mathbb{H}^7}\, h + 7 *_s (v_2^+ \xi_2 x^{\theta_2^+} + S_2(v_2^+)\xi_2 x^{\theta_2^-} + v_3^+ \xi_3 x^{\theta_3^+} + S_3(v_3^+)\xi_3 x^{\theta_3^-}) + O(x^{3+\epsilon}) \\
\mathrm{Tr}_{\mathbb{S}^4}\, g &= \mathrm{Tr}_{\mathbb{S}^4}\, h + 4 *_s (v_2^+ \xi_2 x^{\theta_2^+} + S_2(v_2^+)\xi_2 x^{\theta_2^-} + v_3^+ \xi_3 x^{\theta_3^+} + S_3(v_3^+)\xi_3 x^{\theta_3^-}) + O(x^{3+\epsilon}) \\
g_{(1,1)} &= h_{1,1} + (v_2^+ \xi_2 x^{\theta_2^+} + S_2(v_2^+)\xi_2 x^{\theta_2^-} + v_3^+ \xi_3 x^{\theta_3^+} + S_3(v_3^+)\xi_3 x^{\theta_3^-}) + O(x^{3+\epsilon}) \\
H_{(1,3)} &= -d_H (v_2^+ \xi_2 x^{\theta_2^+} + S_2(v_2^+)\xi_2 x^{\theta_2^-} + v_3^+ \xi_3 x^{\theta_3^+} + S_3(v_3^+)\xi_3 x^{\theta_3^-}) + O(x^{3+\epsilon}) \\
H_{(0,4)} &= 6\,\mathrm{Vol}_{\mathbb{S}^4} + d_s *_s (v_2^+ \xi_2 x^{\theta_2^+} + S_2(v_2^+)\xi_2 x^{\theta_2^-} + v_3^+ \xi_3 x^{\theta_3^+} + S_3(v_3^+)\xi_3 x^{\theta_3^-}) + O(x^{3+\epsilon})
\end{aligned}$$

Then finally using elliptic regularity, we can extend the result to boundary data with Sobolev regularity.

Proposition 2.15. *For any $k > 0$, given boundary data $v_i \in H^k(\mathbb{S}^6)$, the solutions we get is in $H^{s,k}_{e,b}(M;W)$.*

Proof. This follows from the elliptic estimate

$$\|u\|_{H^{s,k}_{e,b}(M;W)} \leq C\|v\|_{H^k(\mathbb{S}^6;V)} + \|Q(u)\|_{H^{s-2,k}_{e,b}(M;W)}.$$

□

2.5 Edge operators

2.5.1 Edge vector fields and edge differential operators

Proposition 2.16. *$H^{s,k}_{e,b}(M)$ is a well-defined space.*

Proof. It is easy to see by using the commutator relation $[\mathcal{V}_e, \mathcal{V}_b] \subset \mathcal{V}_b$. □

Proposition 2.17. *Any m-th order edge operator P maps $H^{s,k}_{e,b}(M)$ to $H^{s-m,k}_{e,b}(M)$, for $m \leq s$.*

Proof. Locally, any m-th order edge operator P can be written in the following form

$$P = \sum_{j+|\alpha|+|\beta| \leq m} a_{j,\alpha.\beta}(x,y,z)(x\partial_x)^j(x\partial_y)^\alpha \partial_z^\beta$$

If we can prove for m=1, P maps $H^{s,k}_{e,b}(M)$ to $H^{s-1,k}_{e,b}(M)$, then by induction, we can prove for any m. Therefore we restrict to the case $m = 1$.

We just need to check that, for a function $u \in H^{s,k}_{e,b}(M)$, Pu satisfies

$$V_e^i Pu \in H^k_b(M), 0 \leq i \leq s-1.$$

The we prove the proposition by induction on k. For k=1 case, since a boundary vector field $V \in \mathcal{V}_b(M)$ satisfies the commutator relation $VP = PV + [V, P]$ where the Lie bracket $[V, P] \in \mathcal{V}_b$, then

$$VP(u) = PV(u) + V_b(u)$$

by definition of $u \in H^{s,k}_{e,b}$, both $V(u)$ and $V_b(u)$ are in $H^s_e(M)$, therefore $PV(u) \in H^{s-1}_e(M)$.

If it holds for $k-1$, then by the relation

$$V_b^k P(u) = V_b^{k-1} PV_b(u) + V_b^k(u),$$

since $V_b(u) \in H^{s,k-1}_{e,b}$ and from induction assumption $PV_b(u) \in H^{s-1,k-1}_{e,b}$, therefore the first term $V_b^{k-1}PV_b(u) \in H_e^{s-1}(M)$, and the second term is in H_e^s by definition. Therefore $Pu \in H_{e,b}^{s-1,k-1}$, which completes the induction. □

2.5.2 Hybrid Sobolev space

Proposition 2.18. *For k large enough and $r \geq -3$, $x^r H_{e,b}^{s,k}(M)$ is an algebra.*

Proof. We first prove that, for the case $r = -3$, the boundary Sobolev space $x^{-3}H_b^k$ is an algebra for large k. Working in the upper half plane model with coordinates $(x, y_1, ...y_n, z)$. For any element $f \in x^{-3}H_b^k(M)$, by definition, its Sobolev norm is

$$\int |V_b^k(x^3 f)|^2 x^{-7} dxdydz$$

Since the commutator $[V_b, x^3]f \subset \{x^3 f\}$, therefore the definition of the Sobolev norm is the same as

$$\int |x^3(V_b^k f)|^2 x^{-7} dxdydz$$

If we do a coordinate transformation to change the problem back to $\mathbb{R}^n$: let $\rho = ln(x)$, then $x\partial_x = \partial_\rho$. Therefore under the new coordinates, the boundary vector fields are spanned by $(\partial_\rho, \partial_y, \partial_z)$. Let F be the function after coordinate transformation

$$F(\rho, y, z) = f(e^\rho, y, z)$$

then from the discussion above we can see

$$\|f\|^2_{x^{-3}H_b^k} = \int |x^3(V_b^k f)|^2 x^{-7} dxdydz = \int |V_b^k F|^2 d\rho dydz < \infty$$

which means $F \in H^k(\mathbb{R}^n)$. From [], the usual Sobolev space in $\mathbb{R}^n$ is closed under multiplication if and only if $k > \frac{n}{2}$. Therefore, take two elements $f, g \in x^{-3}H_b^k(M)$, then the corresponding functions in $\mathbb{R}^n$ satisfy $FG \in H^k(\mathbb{R}^n)$. It follows that $fg \in x^{-3}H_b^k(M)$ by taking the inverse coordinate transformation.

Then it is easy to see that $x^r H_b^k(M)$ is an algebra for $r > -3$. From the result above,

$$
\begin{aligned}
&(x^r H_b^k)\cdot(x^r H_b^k) = x^{3+r}(x^{-3}H_b^k)\cdot x^{3+r}(x^{-3}H_b^k)\\
&\subset\ x^{6+2r}(x^{-3}H_b^k)\ \subset\ x^{3+r}(x^{-3}H_b^k) = x^r H_b^k(M).
\end{aligned}
$$

Now that we proved $H_b^k(M)$ is closed under multiplication, then we want to prove $H_{e,b}^{s,k}(M)$ is also an algebra. For any functions $f, g \in H_{e,b}^{s,k}(M)$, by Leibniz rule,

$$
V_e^j(fg) = \sum_{i=0}^{j} V_e^i(f)V_e^{j-i}(g)
$$

where by assumption, both $V_e^i(f)$ and $V_e^{j-i}(g)$ are in $H_b^k(M)$, therefore their product is also in $H_b^k(M)$ from the above result.Hence we proved $V_e^j(fg) \in H_b^k(M)$ for $0 \leq j \leq s$.which shows $fg \in H_{e,b}^{s,k}(M)$. □

2.6 Computation of the indicial roots

2.6.1 Hodge decomposition

The system contains the following equations, where the (i,j) notations means the splitting of degrees of forms with respect to the product structure of $\mathbb{B}^7 \times \mathbb{S}^4$.

- From the first order equation

$$
(7,1):\quad \begin{gathered} 6d_H *_7 k_{(1,1)} + 3d_S(Tr_{H^7}(k) - Tr_{S^4}(k)) \bigwedge{}^7 V \\ +d_S * H_{(0,4)} + d_H * H_{(1,3)} = 0 \end{gathered} \tag{2.52}
$$

$$
(6,2):\quad d_S * H_{(1,3)} + d_H * H_{(2,2)} + 6d_S *_7 k_{(1,1)} = 0 \tag{2.53}
$$

$$
(5,3):\quad d_S * H_{(2,2)} + d_H * H_{(3,1)} = 0 \tag{2.54}
$$

$$
(4,4):\quad d_S * H_{(3,1)} + d_H * H_{(4,0)} + W \wedge H_{(4,0)} = 0 \tag{2.55}
$$

- From $dH = 0$

$$d_H H_{(0,4)} + d_S H_{(1,3)} = 0 \tag{2.56}$$

$$d_H H_{(1,3)} + d_S H_{(2,2)} = 0 \tag{2.57}$$

$$d_H H_{(2,2)} + d_S H_{(3,1)} = 0 \tag{2.58}$$

$$d_H H_{(3,1)} + d_S H_{(4,0)} = 0 \tag{2.59}$$

$$d_H H_{(4,0)} = 0 \tag{2.60}$$

- From the laplacian:

$$\frac{1}{2}\triangle_s k_{Ij} + \frac{1}{2}\triangle_H k_{Ij} + 6k_{Ij} - 3 *_S H_{(1,3)} = 0 \tag{2.61}$$

$$\frac{1}{2}(\triangle_s + \triangle_H)k_{IJ} - k_{IJ} - 6Tr_S(k)t_{IJ} + Tr_H(k)t_{IJ} + 2H_{(0,4)}t_{IJ} = 0 \tag{2.62}$$

$$\frac{1}{2}(\triangle_S + \triangle_H)k_{ij} + 4k_{ij} + 8Tr_S(k)t_{ij} - H_{(0,4)}t_{ij} = 0 \tag{2.63}$$

2.6.2 Indicial roots

Then we decompose further with respect to Hodge theory on sphere, and compute the indicial roots for each part.

- Harmonic functions on $\mathbb{S}^4$:

  1. Trace-free 2-tensor on H^7, where the equation is

  $$(\mathit{\Delta}_S + \mathit{\Delta}_H - 2)\hat{k}_{IJ} = 0,$$

  and the indicial equation is

  $$(-s^2 + 6s)k_{IJ} = 0.$$

  we have indicial roots

  $$S_1^+ = 0, S_1^- = 6.$$

This corresponds to the perturbation of hyperbolic metric to Poincaré–Einstein metric.

2. Trace-free 2-tensor on S^4, where the equation is

$$\varDelta_S^{rough}\hat{k}_{ij} + \varDelta_H\hat{k}_{ij} + 8\hat{k}_{ij} = 0$$

where indicial equation is

$$(-s^2 + 6s + 8)\hat{k}_{ij} = 0,$$

indicial roots

$$S_2^{\pm} = 3 \pm \sqrt{17}.$$

3. We have

$$d_H * H_{(4,0)} + W \wedge H_{(4,0)} = 0 \tag{2.64}$$

$$d_H H_{(4,0)} = 0 \tag{2.65}$$

The second equation can be deduced from the first one. Since the indicial operator for d_H is

$$I[d](s)w = (-1)^k(s-k)w \wedge dx/x$$

Let

$$H_{(4,0)} = T + dx/x \wedge N$$

be the decomposition with respect to tangential and normal decomposition, then the indicial equations are

$$-(s-3)(*_6 N) \wedge dx/x - 6dx/x \wedge N = 0$$

$$(s-4)T \wedge dx/x = 0$$

where the first equation gives

$$(s-3)*_6 N - 6N = 0$$

i.e. N is an eigenform of $*_6$ and the corresponding indicial roots are

$$s_- = 3-6i, N \in \bigwedge^3(\mathbb{S}^6); *_6N = iN;$$

$$s_+ = 3+6i : N \in \bigwedge^3(\mathbb{S}^6); *_6N = -iN.$$

And plugging into the second equation, we have the vanishing of tangential form

$$T = 0.$$

Therefore the kernel in this case is

$$H_{(4,0)} = dx/x \wedge N, N \in \{\bigwedge^3(\mathbb{S}^6), *_6N = \pm iN\}.$$

- Then we consider Exact 1-form, which includes function/exact 1-form/ coexact 3-form/ exact 4-form on the eigenspace $\lambda = 4(k+1)(k+4)$ starting from $k = 0$.

1. Denote $\tau = \frac{1}{4}Tr_S(k) = \frac{1}{4}t^{ij}k_{ij}, \sigma = \frac{1}{7}Tr_H(k) = \frac{1}{7}t^{IJ}k_{IJ}$ to be the normalized trace, then we have the following equations:

$$\begin{gathered} 6d_H *_H k^{cl}_{(1,1)} + d_S(3Tr_H(k) - 3Tr_S(k))\bigwedge{}^7V \\ +d_S * H^{cl}_{(0,4)} + d_H * H^{cc}_{(1,3)} = 0 \end{gathered} \tag{2.66}$$

$$d_H H^{cl}_{(0,4)} + d_S H^{cc}_{(1,3)} = 0 \tag{2.67}$$

$$d_H H^{cc}_{(1,3)} = 0 \tag{2.68}$$

$$\triangle_s k^{cl}_{(1,1)} + \triangle_H k^{cl}_{(1,1)} + 12k^{cl}_{(1,1)} - 6*_S H^{cc}_{(1,3)} = 0 \tag{2.69}$$

$$\varDelta_S\tau + \varDelta_H\tau + 72\tau - 8*_S H^{cl}_{0,4} = 0 \tag{2.70}$$

$$\varDelta_S\sigma + \varDelta_H\sigma + 12\sigma + 4 *_S H^{cl}_{0,4} - 48\tau = 0 \tag{2.71}$$

First note that 2.68 can be derived from 2.67. Let $H^{cl}_{(0,4)} = d_S\eta$, here η is a (0,3)-form. Then $H^{cc}_{(1,3)} = -d_H\eta$ by 2.68. Let $f = *_S d_S\eta$. Let $k^{cl}_{(1,1)} = d_S w$, w is (1,0)-form. Put it back to 2.66 we get

$$6d_H *_H d_S w + d_S *_H (21\sigma - 12\tau) + *_H d_S *_S d_S\eta - *_S d_H *_H d_H\eta = 0 \tag{2.72}$$

Apply $*_H$ ($*_H^2 = 1$), we get

$$6 *_H d_H *_H d_S w + d_S(21\sigma - 12\tau) + d_S *_S d_S\eta - *_S *_H d_H *_H d_H\eta = 0$$

Then let $\eta = *_S d_S\xi$, ξ be a function, and pull out d_S

$$-6\delta_H w + (21\sigma - 12\tau) - \varDelta_S\xi - \varDelta_H\xi = 0 \tag{2.73}$$

and put the expression to 2.69,

$$\varDelta_S d_S w + \varDelta_H d_S w + 12 d_S w + 6 *_S d_H *_S d_S\xi = 0$$

Apply δ_H and pull out d_S

$$\varDelta_S\delta_H w + \varDelta_H\delta_H w + +12\delta_H w + 6\varDelta_H\xi = 0 \tag{2.74}$$

Now 2.70 becomes

$$\varDelta_S\tau + \varDelta_H\tau + 72\tau + 8\varDelta_S\xi = 0 \tag{2.75}$$

And 2.71 is

$$\varDelta_S\sigma + \varDelta_H\sigma + 12\sigma - 4\varDelta_S\xi - 48\tau = 0 \tag{2.76}$$

Putting the above four equations together, and suppose the eigenvalue of

Δ_S is λ, we get

$$\begin{pmatrix} 12+\lambda+\Delta_H & -48 & -4\lambda & 0 \\ 0 & 72+\lambda+\Delta_H & 8\lambda & 0 \\ 21 & -12 & -\lambda-\Delta_H & -6 \\ 0 & 0 & 6\Delta_H & 12+\lambda+\Delta_H \end{pmatrix} \begin{pmatrix} \sigma \\ \tau \\ \xi \\ \delta_H w \end{pmatrix} = 0$$

The determinant, after putting in the indicial operator of Δ_H, is

$$\begin{aligned} &\lambda^4 - 4S^2\lambda^3 + 24S*\lambda^3 - 90\lambda^3 + 6S^4\lambda^2 - 72S^3\lambda^2 \\ &+ 342S^2\lambda^2 - 756S*\lambda^2 + 1152\lambda^2 - 4S^6\lambda + 72S^5\lambda - 414S^4\lambda \\ &+ 648S^3\lambda + 1152S^2\lambda - 3024S*\lambda + 10368\lambda \\ &+ S^8 - 24S^7 + 162S^6 + 108S^5 - 6192S^4 \\ &+ 31536S^3 - 33696S^2 - 155520S = 0 \end{aligned} \tag{2.77}$$

Putting the lowest two eigenvalues for closed 1-form, we get the following two pairs of roots: for $\lambda = 16$ the indicial roots are $\theta_2 = 3 \pm i\sqrt{21116145}/1655$. with kernel

$$\xi_{16} \in \bigwedge_{\lambda=16}^{cl}(S)$$

which is the closed 1-form on 4-sphere with eigenvalue 16. and the other pair is for $\lambda = 40$ then

$$\theta_3 = 3 \pm i3\sqrt{582842}/20098,$$

with kernel

$$\xi_{40} \in \bigwedge_{\lambda=40}^{cl}(S).$$

2. We have

$$d_S * H^{cl}_{(3,1)} + d_H * H^{cc}_{(4,0)} + 6^4 V \wedge H^{cc}_{(4,0)} = 0, \tag{2.78}$$

$$d_H H^{cl}_{(3,1)} + d_S H^{cc}_{(4,0)} = 0. \tag{2.79}$$

Let

$$H^{cl}_{(3,1)} = d_S \eta$$

where η is (3,0), put into second equation to get

$$H^{cc}_{(4,0)} = -d_H \eta$$

Put everything back to first equation, we get

$$d_S * d_S \eta - d_H * d_H \eta - 6^4 V \wedge d_H \eta = 0.$$

Apply $*_S$, and note $*_S^2 = (-1)^{k(4-k)} = 1$, $\delta_S = (-1)^{4(k+1)+1} *_S d_S *_S = - *_S d_S *_S$, $\varDelta_S = d\delta + \delta d$,

$$*_H(-\delta_S) d_S \eta - d_H *_H d_H \eta + d_H \eta = 0$$

Then apply $*_H$, note $(*_H)^2 = 1$, get

$$-\varDelta_S \eta - *_H d_H *_H d_H \eta + 6 *_H d_H \eta = 0.$$

Let $\varDelta_S \eta = \lambda \eta$

$$-\lambda \eta - \varDelta_H \eta + 6 *_H d_H \eta = 0$$

The indicial equation: using $I[d](s)w = (-1)^k (s-k) w \wedge \frac{dx}{x}$,

$$-\lambda \eta + (s-3)^2 \eta + 6(s-3) *_6 \eta = 0$$

that is

$$(s-3)^2 \pm 6i(s-3) - 16 = 0$$

with roots

$$s = 3 \pm \sqrt{7} \pm 3i.$$

- Then we consider the co-exact 1-form, which contains coexact 1-form/ exact

2-form/ coexact 2-form/ exact 3-form for $\lambda = 4(k+2)(k+3)$ starting from $k = 0$.

1. We have

$$6d_H *_H k^{cc}_{(1,1)} + d_H * H^{cl}_{(1,3)} = 0 \tag{2.80}$$

$$d_S * H^{cl}_{(1,3)} + d_H * H^{cc}_{(2,2)} + 6d_S *_H k^{cc}_{(1,1)} = 0 \tag{2.81}$$

$$d_H H^{cl}_{(1,3)} + d_S H^{cc}_{(2,2)} = 0 \tag{2.82}$$

$$\frac{1}{2}\triangle_s k^{cc}_{(1,1)} + \frac{1}{2}\triangle_H k^{cc}_{(1,1)} + 6k^{cc}_{(1,1)} - \frac{1}{2} *_S H^{cl}_{(1,3)} = 0 \tag{2.83}$$

First note that (2.80) can be derived from (2.81) Let $H^{cl}_{(1,3)} = d_s\eta$, where η is (1,2)-form. Then $H^{cc}_{2,2} = -d_H\eta$ from (2.82). Put it to (2.81), $d_S * d_S\eta - d_H * d_H\eta + 6d_S *_H k^{cc}_{1,1} = 0$. Apply $*_S, *_H$, get $-\Delta_S\eta - \Delta_H\eta + 6 *_S d_S k^{cc}_{(1,1)} = 0$. Apply $*_S d_S$ again, get $-\Delta_S(*_S d_S\eta) - \Delta_H(*_S d_S\eta) - 6\Delta_S k^{cc}_{(1,1)} = 0$. Combining with (2.83), and let λ be the eigenvalue for Δ_S on coclosed 1-form, we get

$$\begin{pmatrix} -\lambda - \Delta_H & -6\lambda \\ -1 & \lambda + \Delta_H + 12 \end{pmatrix} \begin{pmatrix} *_S d_S\eta \\ k^{cc}_{(1,1)} \end{pmatrix} = 0$$

The indicial equation is

$$\lambda^2 - (36 + (s-1)(s-5) + s^2 - 6s - 1)\lambda - (s-1)(s-5)(-s^2 + 6s + 1) = 0.$$

With smallest eigenvalue for coclosed 1-form to be $\lambda = 24$, indicial roots are

$$S_3^{\pm} = 3 \pm \sqrt{\pm 3\sqrt{97} + 31}$$

2.

$$d_S * H^{cl}_{(2,2)} + d_H * H^{cc}_{(3,1)} = 0 \tag{2.84}$$

$$d_H H^{cl}_{(2,2)} + d_S H^{cc}_{(3,1)} = 0 \tag{2.85}$$

Apply d_H and d_S to the equations, we have

$$d_H d_S * H^{cl}_{(2,2)} = 0, d_S d_H H^{cl}_{(2,2)} = 0 \tag{2.86}$$

let $H^{cl}_{(2,2)} = d_S \eta$ where η is a coclosed (2,1)-form, Putting it back, and using d_S is an isomorphism, $d_H \eta = -H^{cc}_{(3,1)}$. Then from first equation, $d_S * d_S \eta - d_H * d_H \eta = 0$, which is $- *_H *_S \Delta_S \eta - *_S *_H \Delta_H \eta = 0$ then it requires $\Delta_H \eta = -\lambda \eta$. Putting $\lambda = 4(k+2)(k+3)$, the result is

$$s = 3 \pm \sqrt{17}.$$

Chapter 3

Resolution of the canonical fiber metrics for a Lefschetz fibration

In the setting of complex surfaces, a Lefschetz fibration is a holomorphic map to a curve, generalizing an elliptic fibration in that it has only a finite number of singular points near which it is holomorphically reducible to normal crossing. Donaldson [8] showed that a four-dimensional simply-connected compact symplectic manifold, possibly after stabilization by a finite number of blow-ups, admits a Lefschetz fibration, in an appropriately generalized sense, over the sphere; Gompf [13] showed the converse. The reader is referred to the book of Gompf and Stipsicz [14] for a description of the important role played by Lefschetz fibrations in the general theory of 4-manifolds.

To cover these cases we consider a compact connected almost-complex 4-manifold M and a smooth map, with complex fibers, to a Riemann surface Z

$$M \xrightarrow{\psi} Z. \tag{3.1}$$

We then require that this map be pseudo-holomorphic, have surjective differential outside a finite set $F \subset M$, on which ψ is injective, so $\psi : F \longleftrightarrow S \subset M$, and near each of these singular points be reducible to the normal crossing, or plumbing variety, model (3.2) below.

A curve of genus g with b punctures is stable if its automorphism group is finite, which is the case when $3g-3+b>0$. In this paper we discuss Lefschetz fibrations with regular fibers having genus $g>1$ and hence stable. All fibers carry a unique metric of curvature -1, for the singular fibers with cusp points replacing the nodes. In view of uniqueness and stability, these metrics necessarily vary smoothly near a regular fiber. We discuss here the precise uniform behavior of this family of metrics near the singular fibers, showing that in terms of appropriate (logarithmic) resolutions, of both the total and parameter spaces, to manifolds with corners the resulting fiber metric is polyhomogeneous and more particularly log-smooth, i.e. essentially smooth except for the appearance of logarithmic terms in the expansions at boundary surfaces. This refines a result of Obitsu and Wolpert [39] who gave the first two terms in the expansion. In a forthcoming paper the universal case of the Deligne-Mumford compactification of the moduli space of Riemann surfaces, also treated by Obitsu and Wolpert, will be discussed.

The local model for degeneration for the complex structure on a Riemann surface to a surface with a node is the 'plumbing variety' with its projection to the parameter space. We add boundaries, away from the singularity at the origin, to make this into a manifold with corners:

$$\begin{gathered} P=\Big\{(z,w)\in\mathbb{C}^2;\ \exists\, t\in\mathbb{C},\ zw=t,\ |z|\le\frac{3}{4},\ |w|\le\frac{3}{4},\ |t|\le\frac{1}{2}\Big\} \\ P\xrightarrow{\phi}\mathbb{D}_{\frac{1}{2}}=\Big\{t\in\mathbb{C};|t|\le\frac{1}{2}\Big\}. \end{gathered} \tag{3.2}$$

Thus near each point of F we require that ψ can be reduced to ϕ in (almost) holomorphic coordinates in M and Z.

A (real) manifold with corners M has a principal ideal $\mathcal{I}_F\subset\mathcal{C}^\infty(M)$ corresponding to each boundary hypersurface (by assumption embedded and connected) generated by a boundary defining function $\rho_F\ge 0$ with $F=\{\rho_F=0\}$ and $d\rho_F\neq 0$ on F. A smooth map between manifolds with corners $f:M\longrightarrow Y$ is an interior b-map if each of these ideals on Y pulls back to non-trivial finite products of the corresponding ideals on M, it is b-normal if there is no common factor in these product decompositions – this is always the case here since the range space is a manifold with boundary. Such a map is a b-fibration if in addition every smooth vector field tangent to all boundaries on Y is locally (and hence globally) f-related to such a vector field on M; it is then surjective. There is a slightly weaker notion than a manifold with corners, a *tied* manifold, which has the same local structure but in which the boundary hypersurfaces need not be embedded, meaning that transversal self-intersection is allowed. This arises below, although not in any essential way. There is still a principal ideal associated to each boundary hypersurface and the notions above carry over.

The assumptions above mean that each singular fiber of ψ has one singular point at which it has a normal crossing in the (almost) complex sense as a subvariety of M. The first step in the resolution is the blow up, in the real sense, of the singular fibers; this is well-defined in view of the transversality of the self-instersection but results in a tied manifold since the boundary faces are not globally embedded. The second step is to replace the $\mathcal{C}^\infty$ structure by its logarithmic weakening, i.e. replacing each (local) boundary defining function x by

$$\operatorname{ilog} x = (\log x^{-1})^{-1}.$$

This gives a new tied manifold mapping smoothly to the previous one by a homeomorphism. These two steps can be thought of in combination as the 'logarithmic blow up' of the singular fibers. The final step is to blow up the corners, of codimension two, in the preimages of the singular fibers. This results in a manifold with corners, M_{mr}, with the two boundary hypersurfaces denoted B_{I}, resolving the singular fiber, and B_{II} arising at the final stage of the resolution. The parameter space Z is similarly resolved to a manifold with corners by the logarithmic blow up of each of the singular points.

It is shown below that the Lefschetz fibration lifts to a smooth map

$$M_{\mathrm{mr}} \xrightarrow{\psi_{\mathrm{mr}}} Z_{\mathrm{mr}} \tag{3.3}$$

which is a b-fibration. In particular it follows from this that smooth vector fields on

M_{mr} which are tangent to all boundaries and to the fibers of ψ_{mr} form the sections of a smooth vector subbundle of ${}^{\mathrm{b}}TM_{\mathrm{mr}}$ of rank two. The boundary hypersurface B_{II} has a preferred class of boundary defining functions, an element of which is denoted ρ_{II}, arising from the logarithmic nature of the resolution, and this allows a Lie algebra of vector fields to be defined by

$$V \in \mathcal{C}^\infty(M_{\mathrm{mr}}; {}^{\mathrm{b}}TM_{\mathrm{mr}}),\ V\psi^*\mathcal{C}^\infty(Z_{\mathrm{mr}}) = 0,\ V\rho_{\mathrm{II}} \in \rho_{\mathrm{II}}^2\mathcal{C}^\infty(M_{\mathrm{mr}}). \tag{3.4}$$

The possibly singular vector fields of the form $\rho_{\mathrm{II}}^{-1}V$, with V as in (3.4), also form all the sections of a smooth vector bundle, denoted ${}^{L}TM_{\mathrm{mr}}$. This vector bundle inherits a complex structure and hence has a smooth Hermitian metric, which is unique up to a positive smooth conformal factor on M_{mr}. The main result of this paper is:

Theorem 3.1. *The fiber metrics of fixed constant curvature on a Lefschetz fibration, in the sense discussed above, extend to a continuous Hermitian metric on* ${}^{L}TM_{\mathrm{mr}}$ *which is related to a smooth Hermitian metric on this complex line bundle by a log-smooth conformal factor.*

The notion of log-smoothness here, for a function, is the same as polyhomogeneous conormality with non-negative integral powers and linear multiplicity of slope one. Conormality in this context for $f : M_{\mathrm{mr}} \longrightarrow \mathbb{R}$ can be interpreted as the 'symbol estimates' that

$$f \in \mathcal{A}(M_{\mathrm{mr}}) \Longleftrightarrow \mathrm{Diff}^*_{\mathrm{b}}(M_{\mathrm{mr}})f \subset L^\infty(M_{\mathrm{mr}}) \tag{3.5}$$

which in fact implies that the space of these functions is stable under the action, $\mathrm{Diff}^*_{\mathrm{b}}(M_{\mathrm{mr}})\mathcal{A}(M_{\mathrm{mr}}) \subset \mathcal{A}(M_{\mathrm{mr}})$. Polyhomogeneity means the existence of appropriate expansions at the boundary. On a manifold with boundary, M, log-smoothness of a conormal function $f \in \mathcal{A}(M)$ means the existence of an expansion at the boundary, generalizing the Taylor series of a smooth function, so for any product decomposition near the boundary with boundary defining function x, there exist coefficients $a_{j,k} \in$

$\mathcal{C}^\infty(\partial M)$, $j \geq 0$, $j \geq k \geq 0$ such that for any finite N,

$$f - \sum_{j \leq N, 0 \leq k \leq j} a_{j,k} x^j (\log x)^k \in x^N \mathcal{A}([0,1) \times \partial M), \ \forall \ N. \tag{3.6}$$

We denote the linear space of such functions $\mathcal{C}^\infty_{\log}(M)$, it is independent of choices.

In the case of a manifold with corners the definition may be extended by iteration of boundary codimension. Thus $f \in \mathcal{C}^\infty_{\log}(M_{\mathrm{mr}})$ if for any product decompositions of M_{mr} near the two boundaries there are corresponding coefficients $a_{j,k,b} \in \mathcal{C}^\infty_{\log}(B_b)$, $b = \mathrm{I}, \mathrm{II}$, such that

$$f - \sum_{j \leq N, 0 \leq k \leq j} a_{j,k,b} x_b^j (\log x_b)^k \in x_b^N \mathcal{A}([0,1) \times B_b), \ b = \mathrm{I}, \mathrm{II}, \ \forall \ N. \tag{3.7}$$

There are necessarily compatibility conditions between the two expansions at the corners, $B_{\mathrm{I}} \cap B_{\mathrm{II}}$, and together they determine f up to a smooth function on M_{mr} vanishing to infinite order on both boundaries. In this sense the conformal factor in the main result above is 'essentially smooth'.

In the model setting, (3.2), there is an explicit family of fiber metrics, the 'plumbing metric', of curvature -1,

$$\begin{aligned} g_P &= \Big(\frac{\pi \log|z|}{\log|t|} \csc \frac{\pi \log|z|}{\log|t|}\Big)^2 ds_0^2, \\ g_0 &= \Big(\frac{|dz|}{|z| \log|z|}\Big)^2. \end{aligned} \tag{3.8}$$

This metric can be extended ('grafted' as in [39]) to give an Hermitian metric on ${}^L TM_{\mathrm{mr}}$ which has curvature R equal to -1 near B_{II} and to second order at B_{I}. We prove the Theorem above by constructing the conformal factor e^{2f} for this metric which satisfies the curvature equation, ensuring that the new metric has curvature -1 :

$$(\Delta + 2) f + (R + 1) = -e^{2f} + 1 + 2f = O(f^2). \tag{3.9}$$

This equation is first solved in the sense of formal power series (with logarithms) at both boundaries, B_{I} and B_{II}, which gives us an approximate solution f_0 with

$$-\Delta f_0 = R + e^{2f_0} + g,\ g \in s_t^\infty \mathcal{C}^\infty(M_{\mathrm{mr}}).$$

Then a solution $f = f_0 + \tilde{f}$ to (3.9) amounts to solving

$$\tilde{f} = -(\Delta + 2)^{-1} \Big(2\tilde{f}(e^{2f_0} - 1) + e^{2f_0}(e^{2\tilde{f}} - 1 - 2\tilde{f}) - g \Big) = K(\tilde{f}).$$

Here the non-linear operator K is at least quadratic in $\tilde{f}$ and the boundedness of $(\Delta + 2)^{-1}$ on $\rho_{\mathrm{II}}^{-\frac{1}{2}} H_{\mathrm{b}}^M(M_{\mathrm{mr}})$ for all M allow the Inverse Function Theorem to be applied to show that $\tilde{f} \in s_t^\infty \mathcal{C}^\infty(M_{\mathrm{mr}})$ and hence that f itself is log-smooth.

In §3.1 the model space and metric are analysed and in §3.2 the global resolution is described and the proof of the Theorem above is outlined. The linearized model involves the inverse of $\Delta + 2$ for the Laplacian on the fibers and the uniform behavior, at the singular fibers, of this operator is explained in §3.3. The solution of the curvature problem in formal power series is discussed in §3.4 and using this the regularity of the fiber metric is shown in §3.5.

3.1 The plumbing model

We start with a description of the real resolution of the plumbing variety, given in (3.2), and the properties of the fiber metric, (3.8), on the resolved space. As noted above there are three steps in this resoluton, first the fiber complex structure is resolved, in a real sense, then two further steps are required to resolve the fiber metric.

The plumbing variety itself is smooth with z and w global complex coordinates – it is the model singular fibration ϕ which is to be 'resolved' in the real sense. The fibers above each $t \neq 0$ are annuli

$$\{|t| \le |z| \le \tfrac{3}{4}\} \xrightarrow{w=t/z} \{|t| \le |w| \le \tfrac{3}{4}\} \tag{3.10}$$

whereas the singular fiber above $t = 0$ is the union of the two discs at $z = 0$ and $w = 0$ identified at their origins

$$\phi^{-1}(0) = \left\{|z| \le \frac{3}{4}\right\} \cup \left\{|w| \le \frac{3}{4}\right\} / (\{z = 0\} \sim \{w = 0\}). \tag{3.11}$$

Note that the differential of ϕ vanishes at the singular point $z = w = 0$ so any smooth vector field on the range which lifts under it, i.e. is ϕ-related to a smooth vector field on P, vanishes at $t = 0$. Conversely, $t\partial_t$ is ϕ-related to both $z\partial_z$ and $w\partial_w$ whereas the vector field

$$V = z\partial_z - w\partial_w \tag{3.12}$$

annihilates $\phi^* t$ and so is everywhere tangent to the fibers of ϕ.

The first step in the resolution of $\phi : P \longrightarrow \mathbb{D}_{\frac{1}{2}}$ consists in passing to the commutative square

$$\begin{array}{ccc} P_{\overline{\partial}} & \xrightarrow{\phi_{\overline{\partial}}} & [\mathbb{D}_{\frac{1}{2}}, 0] \\ \downarrow & & \downarrow \\ P & \xrightarrow[\phi]{} & \mathbb{D}_{\frac{1}{2}}. \end{array} \tag{3.13}$$

Here $[\mathbb{D}_{\frac{1}{2}}, 0]$ is the space obtained by real blow up of the origin in the disk, which can be realized globally as

$$[\mathbb{D}_{\frac{1}{2}}, 0] \simeq \left[0, \frac{1}{2}\right] \times \mathbb{S} \ni (r, \theta) \longmapsto t = re^{i\theta} \in \mathbb{D}_{\frac{1}{2}} \tag{3.14}$$

if $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$. As a real blow-up $[\mathbb{D}_{\frac{1}{2}}, 0]$ is a well-defined manifold with boundary and any diffeomorphism of $\mathbb{D}_{\frac{1}{2}}$ fixing the origin lifts to a global diffeomorphism. The complex structure on $\mathbb{D}_{\frac{1}{2}}$ lifts to a complex structure on ${}^{\mathrm{b}}T[\mathbb{D}_{\frac{1}{2}}, 0]$ generated by $t\partial_t = r\partial_r + i\partial_\theta$ in terms of (3.14).

Proposition 3.1. *The space*

$$P_{\overline{\partial}} = [P; \{z = 0\} \cup \{w = 0\}], \tag{3.15}$$

obtained by the real blow-up of the two normally-intersecting divisors forming the *singular fiber of ϕ, gives a commutative diagram* (3.13) *in which $\phi_{\overline{\partial}}$ is a b-fibration with*

$$\phi_{\overline{\partial}}^* \mathcal{I}_{\overline{\partial}} = \mathcal{I}_{I,L} \mathcal{I}_{I,R} \tag{3.16}$$

where $\mathcal{I}_{I,L}$ and $\mathcal{I}_{I,R}$ correspond to the two boundary components introduced by the blow-up, forming the proper tranforms of $z = 0$ and $w = 0$ respectively.

Proof. The two divisors forming the singular fiber $\phi^{-1}(0)$ are each contained in a product product neighborhood $\mathbb{D}_{\frac{1}{2}} \times \mathbb{D}_{\frac{3}{4}} \subset P$ and $\mathbb{D}_{\frac{3}{4}} \times \mathbb{D}_{\frac{1}{2}} \subset P$. The transversality of their intersection is clear and it follows that the blow-up is well-defined independently of order with the new front faces being

$$B_{\mathrm{I,L}} = \mathbb{S} \times [\mathbb{D}_{\frac{3}{4}}, \{0\}] \subset P_{\overline{\partial}}, \ B_{\mathrm{I,R}} = [\mathbb{D}_{\frac{3}{4}}, \{0\}] \times \mathbb{S} \subset P_{\overline{\partial}}. \tag{3.17}$$

Here each of the blown up disks corresponds to the introduction of polar coordinates, so $r_z = |z|$ is a defining function (globally) for $B_{\mathrm{I,L}}$ and $r_w = |w|$ for $B_{\mathrm{I,R}}$. Since $r_t = |t|$ is a defining function for the blown-up disk in the range and

$$r_t = r_z r_w \tag{3.18}$$

the b-fibration condition follows from the behaviour of the corresponding angular variables

$$e^{i\theta_t} = e^{i\theta_z} e^{i\theta_w}. \tag{3.19}$$

As a compact manifold with corners, $P_{\overline{\partial}}$ is globally the product of an embedded manifold in $\mathbb{R}^2$ and a 2-torus

$$P_{\overline{\partial}} = \left\{ (r_z, r_w) \,\middle|\, 0 \leq r_z, r_w \leq \frac{3}{4}, \ r_z r_w \leq \frac{1}{2} \right\} \times \mathbb{S}_z \times \mathbb{S}_w. \tag{3.20}$$

□

This first step in the resolution resolves the complex structure in a real sense. In particular the vector fields tangent to the fibers of $\phi_{\overline{\partial}}$ and to the boundaries form all the sections of a subbundle of ${}^{\mathrm{b}}TP_{\overline{\partial}}$ which has a complex structure, spanned by the lift of the single vector field (3.12).

Although the complex structure is effectively resolved, the plumbing metric in (3.8) is not. That g_P has curvature -1 on the fibers, away from the singular point, can be seen by changing variables to $s = \log r$, $r = r_z$ and $\theta = \theta_z$ in terms of which

$$g_P = \Big(\frac{\pi/\log|t|}{\sin(\pi s/\log|t|)}\Big)^2(ds^2 + d\theta^2).$$

It then follows from the standard formula for the Gauss curvature that

$$R = -\frac{1}{2\sqrt{fg}}\left(\partial_r\Big(\frac{\partial_r g}{\sqrt{fg}}\Big) + \partial_\theta\Big(\frac{\partial_\theta f}{\sqrt{fg}}\Big)\right) = -1.$$

In view of the coefficients in g_P it is natural to introduce the inverted logarithms of the new boundary defining functions, so replacing the radial by the logarithmic blow-up. Thus

$$s_z = \operatorname{ilog} r_z = \frac{1}{\log\frac{1}{r_z}},\ s_w = \operatorname{ilog} r_w \tag{3.21}$$

become new boundary defining functions in place of r_z and r_w. The space with this new $\mathcal{C}^\infty$ structure can be written

$$[P; \{z = 0\}_{\log} \cup \{w = 0\}_{\log}]. \tag{3.22}$$

However, even after this second step, the fiber metric does not have smooth coefficients:

$$g_P = \frac{\pi^2 s_t^2}{\sin^2(\frac{\pi s_t}{s_w})}\left(\frac{ds_w^2}{s_w^4} + d\theta_w^2\right).$$

Indeed $s_t = \frac{s_z s_w}{s_z+s_w}$ is not a smooth function on the space (3.22).

The final part of the metric resolution is to blow up, radially, the corner formed by the intersection of the two logarithmic boundary faces

$$P_{\mathrm{mr}} = [[P; \{z = 0\}_{\log} \cup \{w = 0\}_{\log}]; \{s_z = s_w = 0\}]. \tag{3.23}$$

In terms of the presentation (3.20) this preserves the torus factor and replaces the 2-manifold with corners by a new one with more smooth functions and an extra boundary hypersurface.

Proposition 3.2. *The model Lefschetz fibration* ϕ *lifts to a b-fibration* ϕ_{mr} *giving a commutative diagram*

$$\begin{array}{ccc} P_{\mathrm{mr}} & \xrightarrow{\phi_{\mathrm{mr}}} & [\mathbb{D}_{\frac{1}{2}}; \{0\}_{\log}] \\ \downarrow & & \downarrow \\ P & \xrightarrow[\phi]{} & \mathbb{D}_{\frac{1}{2}}. \end{array} \tag{3.24}$$

Proof. The radial variables on the spaces $P_{\overline{\partial}}$ and $[\mathbb{D}_{\frac{1}{2}}, \{0\}]$ are related by

$$|t| = |z||w| \Longrightarrow s_t = \frac{s_z s_w}{s_z + s_w}, \ s_t = \mathrm{ilog}\,|t| \tag{3.25}$$

so ϕ does not lift to be smooth. However, consider the further introduction of the radial variable $R = (s_z^2 + s_w^2)^{\frac{1}{2}}$ and the smooth defining functions $R_z = s_z/R$, $R_w = s_w/R$ for the lifts of the two boundary hypersurfaces. Then

$$s_t = \frac{R_z R R_w}{R_z + R_w} \tag{3.26}$$

which is smooth since R_z and R_w have disjoint zero sets. It follows that ϕ lifts to a b-fibration as in (3.24) under which the boundary ideal lifts to the product of the three ideals

$$\phi_{\mathrm{mr}}^* \mathcal{I}_{s_t} = \mathcal{I}_{R_z} \mathcal{I}_R \mathcal{I}_{R_w}. \tag{3.27}$$

□

The generator V, in (3.12), of the fiber tangent space of ϕ lifts to $P_{\overline{\partial}}$ as

$$V = r_z \partial_{r_z} - r_w \partial_{r_w} - i\partial_{\theta_z} + i\partial_{\theta_w}$$

in terms of the coordinates in (3.19) and (3.18). Under the introduction of the logarithmic variables in (3.21) it further lifts to

$$V = s_z^2\partial_{s_z} - s_w^2\partial_{s_w} - i\partial_{\theta_z} + i\partial_{\theta_w}.$$

In a neighborhood of the the lift of the face $s_z = 0$ to P_{mr} the variables s_w (defining the new front face) and $\rho_z = s_z/s_w \in [0,\infty)$ (defining the lift of $s_z = 0$) are valid and

$$V = -s_w(s_w\partial_{s_w} - \rho_z\partial_{\rho_z} - \rho_z^2\partial_{\rho_z}) - i\partial_{\theta_z} + i\partial_{\theta_w}. \tag{3.28}$$

Reviewing the three steps in the construction of P_{mr}, notice that the two holomorphic defining functions z and w are well-defined up to constant multiples and addition of (holomorphic) terms $O(|z|^2)$ and $O(|w|^2)$ respectively. Under these two changes, the logarithmic variables s_z change to $s_z + s_z^2 G$ with $G \in \mathcal{C}^\infty(P_{\mathrm{mr}})$ smooth. The same is true of s_w so it follows that the radial variable

$$R = (s_z^2 + s_w^2)^{1/2} \in \mathcal{C}^\infty(P_{\mathrm{mr}}), \tag{3.29}$$

which defines the front face, is also uniquely defined up to an additive term vanishing quadratically there. This determines a 'cusp' structure at B_{II} and from (3.28) we conclude that

Lemma 3.1. *The vector field $R^{-1}V$ on P_{mr} spans a smooth complex line bundle, ${}^{L}TP_{\mathrm{mr}}$ over P_{mr} with underlying real plane bundle having smooth sections precisely of the form $R^{-1}W$ where W is a smooth vector field tangent to the boundaries, to the fibers of ϕ_{mr} and satisfying $WR = O(R^2)$ at $R = 0$.*

It is natural to consider this bundle, precisely because

Lemma 3.2. *The plumbing metric defines an Hermitian metric on ${}^{L}TP_{\mathrm{mr}}$.*

Proof. On P_{mr}, in a neighborhood of the lift of $\{s_z = 0\}$ as discussed above,

$$s_t = \mathrm{ilog}\,|t| = \frac{s_z s_w}{s_z + s_w} = \frac{\rho_z s_w}{1+\rho_z},\ \frac{\log|z|}{\log|t|} = \frac{1}{1+\rho_z}$$

so the fiber metric lifts to

$$g = \frac{\pi^2 s_t^2}{\sin^2(\frac{\pi s_t}{s_w})} \left(\frac{ds_w^2}{s_w^4} + d\theta_w^2 \right) = \frac{\pi^2 s_t^2}{\sin^2(\frac{\pi}{1+\rho_z})} \left(\frac{d\rho_z^2}{s_t^2(1+\rho_z)^4} + d\theta_z^2 \right). \tag{3.30}$$

This is Hermitian and the length of V relative to it is a smooth positive multiple of R^2. □

3.2 Global resolution and outline

It is now straightforward to extend the resolution of the plumbing variety to a global resolution of any Lefschetz fibration as outlined in the Introduction. By hypothesis, the singular fibers of a Lefschetz fibration ψ, as in (3.1), are isolated and each contains precisely one singular point. Near the singular point the map ψ is reduced to ϕ by local complex diffeomorphisms. Thus each singular fiber is a connected compact real manifold of dimension two with a trasversal self-intersection. The real blow-up of such a submanifold is well-defined, since it is locally well-defined away from the self-intersection and well-defined near the intersection in view of the transversality. Thus

$$M_{\overline{\partial}} = [M, \phi^{-1}(S)] \xrightarrow{\psi_{\overline{\partial}}} [Z, S] \tag{3.31}$$

reduces to $\phi_{\overline{\partial}}$ near the preimage of the finite singular set $F \subset M$. Similarly, the logarithmic step can be extended globally since away from the singular set it corresponds to replacing $|z|$, by $\operatorname{ilog}|z|$. Here z is a local complex defining function with holomorphic differential along the singular fiber. Finally, the third step is within the preimage of the set of the singular points and so is precisely the same as for the plumbing variety.

Thus the resolved space M_{mr} with its global b-fibration (3.3) is well-defined as is the Hermitian bundle ${}^{L}TM_{\mathrm{mr}}$ which reduces to ${}^{L}TP_{\mathrm{mr}}$ near the singular points and is otherwise the bundle of fiber tangents to M_{mr} with its inherited complex structure.

To arrive at the description of the constant curvature fiber metric, as an Hermitian metric on ${}^{L}TM_{\mathrm{mr}}$ we start with the "grafting" construction of Obitsu and Wolpert which we interpret as giving a good initial choice of Hermitian metric. Namely choose any smooth Hermitian metric h_0 on ${}^L TM_{\mathrm{mr}}$; from Lemma 3.2

$$g_{\mathrm{Pl}} = e^{f_{\mathrm{Pl}}} h_0 \text{ near } B_{\mathrm{II}},\ f_{\mathrm{Pl}} \text{ smooth.} \tag{3.32}$$

Away from the singular set, near the singular fiber, ψ is a fibration in the real sense. Thus, it has a product decomposition, with the fibration ψ the projection, and this can be chosen to be consistent with the product structure on P away from the singular point. Then the complex structure on the fibers is given by a smoothly varying tensor J. The constant curvature metric g_0 on the resolved singular fiber may therefore be extended trivially to a metric on the fibers nearby, away from the singular points. This has non-Hermitian part vanishing at the singular fiber, so removing this gives a smooth family of Hermitian metrics reducing to g_0 and so with curvature equal to -1 at the singular fiber. After blow-up this remains true since the regular part of the singular fiber is replaced by a trivial circle bundle over it. On the introduction of the logarithmic variables in the base and total space, the curvature of this smooth family, g_{I}, is constant to infinite order at the singular fiber since it is equal to the limiting metric g_0 to infinite order. Comparing g_{I} to the chosen Hermitian metric gives a conformal factor $g_{\mathrm{I}} = e^{f_{\mathrm{I}}} h,\ f_{\mathrm{I}} \in \mathcal{C}^\infty(N)$ where N is a neighborhood of B_{I} excluding a neighborhood of B_{II}. Moreover, g_{Pl} is also equal to the trivial extension of g_0 to second order in a compatible trivialization so the two conformal factors

$$f_{\mathrm{I}} = f_{\mathrm{Pl}} \text{ to second order} \tag{3.33}$$

in their common domain of definition.

The grafting construction of Obitsu and Wolpert interpreted in this setting is then to choose a cutoff $\chi \in \mathcal{C}^\infty(M_{\mathrm{mr}})$ equal to 1 in a neighborhood of B_{II} and supported near it and to set

$$h = e^{\chi f_{\mathrm{Pl}} + (1-\chi) f_{\mathrm{I}}} h_0. \tag{3.34}$$

It follows from the discussion above that h is a smooth Hermitian metric on ${}^L TM_{\mathrm{mr}}$

near the preimage of the singular fibers and that its curvature

$$R(h) = \begin{cases} -1 \text{ near } B_{\mathrm{II}} \\ -1 + O(s_t^2) \text{ near } B_{\mathrm{I}}. \end{cases} \tag{3.35}$$

We therefore use this in place of the initial choice of Hermitian metric.

Let g be the unique Hermitian constant curvature metric on the regular fibers of ψ, so $g = e^{2f}h$. The curvatures are related by

$$R(g)e^{2f} = \Delta_h f + R(h),$$

which reduces to the curvature equation

$$\Delta f + R(h) = -e^{2f},\ \Delta = \Delta_h. \tag{3.36}$$

The linearization of this equation is

$$(\Delta + 2)f = -(R(h) + 1). \tag{3.37}$$

The uniform invertibility of $\Delta + 2$ with respect to the metric L^2 norm, shown below, implies that (3.36) has a unique small solution for small values of the parameter. The proof of the Theorem in the Introduction therefore reduces to the statement that (3.36) has a log-smooth solution vanishing at the boundary.

3.3 Bounds on $(\Delta + 2)^{-1}$

In the linearization of the curvature equation (3.37), the operator $\Delta + 2$, for the fixed initial choice of smooth fiber hermitian metric, appears. For the Laplacian on a compact manifold, $\Delta + 2$ is an isomorphism of any Sobolev space H^{k+1} to H^{k-1}, in particular this is the case for the map from the Dirichlet space to its dual, corresponding to the case $k = 0$. For a smooth family of metrics on a fibration the family of Dirichlet spaces forms the fiber H^1 space and its dual the fiber H^{-1} space and $\Delta+2$ is again an isomorphism between them. These spaces are modules over the $\mathcal{C}^\infty$ functions of the total space and this, plus a simple commutation argument, shows that in this case of a fibration $\Delta+2$ is an isomorphism for any $k \geq 1$ between the space of functions with up to k derivatives, in all directions, in the Dirichlet domain to the space with up to k derivatives in the dual to the Dirichlet space. In particular it follows from this that $\Delta+2$ is an isomorphism on functions supported away from the boundary:

$$\Delta+2 : \mathcal{C}^\infty_{\mathrm{c}}(M_{\mathrm{reg}}) \longleftrightarrow \mathcal{C}^\infty_{\mathrm{c}}(M_{\mathrm{reg}}),\ M_{\mathrm{reg}} = M_{\mathrm{mr}} \setminus \partial M_{\mathrm{mr}}. \tag{3.38}$$

We extend this result up to the boundary of the resolved space for the Lefschetz fibration in terms of tangential regularity.

Proposition 3.3. *For the Laplacian of the grafted metric*

$$(\Delta+2)^{-1} : \rho_{II}^{-\frac{1}{2}} H^k_{\mathrm{b}}(M_{\mathrm{mr}}) \longrightarrow \rho_{II}^{-\frac{1}{2}} H^k_{\mathrm{b}}(M_{\mathrm{mr}})\ \forall\ k \in \mathbb{N}. \tag{3.39}$$

The main complication in the proof arises from the fact that the Dirichlet space is not a $\mathcal{C}^\infty$ module.

First consider the following analog of Fubini's theorem.

Lemma 3.3. *For the fiber metrics corresponding to an Hermitian metric on* ${}^{L}TM_{\mathrm{mr}}$, *the metric density is of the form*

$$|dg| = \rho_{II}\nu_{\mathrm{b,fib}} \tag{3.40}$$

and the space of weighted L^2 *functions with values in the* L^2 *spaces of the fibers can be realized as*

$$L^2(M_{\mathrm{mr}}; |dg|\phi^*_{\mathrm{mr}}\nu_{\mathrm{b}}(Z_{\mathrm{mr}})) = L^2_{\mathrm{b}}(Z_{\mathrm{mr}}; L^2(|dg|)) = \rho_{II}^{-\frac{1}{2}} L^2_{\mathrm{b}}(M_{\mathrm{mr}}). \tag{3.41}$$

Proof. Away from $B_{\mathrm{II}} \subset M_{\mathrm{mr}}$ the resolved map ψ_{mr} is a fibration, ${}^{L}TM_{\mathrm{mr}}$ is the fiber tangent bundle and the boundary is in the base. Thus (3.40) and (3.41) reduce to the local product decomposition for a fibration and Fubini's Theorem.

It therefore suffices to localize near B_{II} and to consider the plumbing metric since all hermitian metrics on ${}^{L}TM_{\mathrm{mr}}$ are quasi-conformal. The symmetry in z and w means that it suffices to consider the region in which $\rho_z = s_z/s_w$ and s_w are defining functions for the two boundary hypersurfaces B_{I} and B_{II} respectively. The plumbing metric may then be written

$$g = \frac{\pi^2 s_t^2}{\sin^2(\frac{\pi s_t}{s_w})}\left(\frac{ds_w^2}{s_w^4} + d\theta_w^2\right) = \frac{\pi^2 s_t^2}{\sin^2(\frac{\pi}{1+\rho_z})}\left(\frac{d\rho_z^2}{s_t^2(1+\rho_z)^4} + d\theta_z^2\right).$$

Thus the fiber area form,

$$|dg| = \frac{\pi^2 s_t^2}{\sin^2(\frac{\pi}{1+\rho_z})}\frac{d\rho_z}{s_t(1+\rho_z)^2 d\theta_z} = f(\rho_z)\frac{s_t}{\rho_z}\frac{d\rho_z}{\rho_z}d\theta_z = \tilde{f}(\rho_z)s_w\frac{d\rho_z}{\rho_z}d\theta_z,$$

is a positive multiple of $s_w \frac{d\rho_z}{\rho_z} d\theta_z$ from which (3.40) follows.

The identication (3.41) holds after localization away from B_{II} and locally near it

$$||f||^2_{L^2_b(Z_{\mathrm{mr}});L^2(dg))} = \int\int |f|^2|dg|\frac{ds_t}{s_t}d\theta_t = \int_{Z_{\mathrm{mr}}} |f|^2 \rho_{\mathrm{II}}\nu_{\mathrm{b}}.$$

□

Since $(\Delta+2)^{-1}$ is a well-defined bounded operator on the metric L^2 space which depends continuously on the parameter in $Z\setminus S$ with norm bounded by 1/2, it follows from (3.41) that

$$(\Delta+2)^{-1} \text{ is bounded on } \rho_{\mathrm{II}}^{-\frac{1}{2}}L^2_{\mathrm{b}}(M_{\mathrm{mr}}). \tag{3.42}$$

We consider the 'total' Dirichlet space based on this L^2 space – we are free to choose the weighting in the parameter space. Thus, let D be the the completion of the smooth functions on M_{mr} supported in the interior with respect to

$$\|u\|^2_D = \int \left(|d_{\mathrm{fib}}u|^2_g + 2|u|^2\right)|dg|\phi^*\nu_{\mathrm{b}}(Z_{\mathrm{mr}}). \tag{3.43}$$

Note that D depends only on the quasi-isometry class of the fiber Hermitian metric but does depend on the induced fibration of the boundary B_{II}.

The dual space, D', to D as an abstract Hilbert space, may be embedded in the extendible distributions on M_{mr} using the volume form $\phi^*_{\mathrm{mr}}\nu_{\mathrm{b}}|dg|$. As is clear from the discussion below, the image is independent of the choice of, ν_{b}, of a logarithmic area form on Z_{mr} but the embedding itself depends on this choice. Thus, $\tilde{v} \in D'$ is identified as a map $v : \dot{\mathcal{C}}^\infty_c(M_{\mathrm{mr}}) \longrightarrow \mathbb{C}$ by

$$\int v\phi|dg|\phi^*\nu_{\mathrm{b}}(Z_{\mathrm{mr}}) = \tilde{v}(\phi). \tag{3.44}$$

We consider the space of vector fields $\mathcal{W} \subset \rho_{\mathrm{II}}^{-1}\mathcal{V}_{\mathrm{b}}(M_{\mathrm{mr}})$ which are tangent to the fibers of ψ_{mr} and to the fibers of B_{II} and which commute with ∂_{θ_z} and ∂_{θ_w} near B_{II}.

Proposition 3.4. *For the grafted metric*

$$\varDelta + 2 : D \to D' \subset \mathcal{C}^{-\infty}(M_{\mathrm{mr}})$$

is an isomorphism, where the elements of D' are precisely those extendible distributions which may be written as finite sums

$$v = \sum_j W_j u_j,\ W_j \in \mathcal{W},\ u_j \in \rho_{II}^{-\frac{1}{2}} L^2_{\mathrm{b}}(Z_{\mathrm{mr}}) \tag{3.45}$$

and has the injectivity property that

$$u \in \mathcal{C}^{-\infty}(M_{\mathrm{mr}}),\ (\varDelta + 2)u \in D' \Longrightarrow u \in D. \tag{3.46}$$

This result remains true for any Hermitian metric on ${}^LTM_{\mathrm{mr}}$ but is only needed here for the grafted metric which is equal to the plumbing metric near B_{II}.

Proof. Although defined above by completion of the space of smooth functions supported away from the boundary of M_{mr} with respect to the norm (3.43) the space D

can be identified in the usual way with the subspace of $\mathcal{C}^{-\infty}(M_{\mathrm{mr}})$ consisting of those

$$u \in \rho_{\mathrm{II}}^{-\frac{1}{2}} L^2_{\mathrm{b}}(M_{\mathrm{mr}}) \text{ s.t. } \mathcal{W} \cdot u \subset \rho_{\mathrm{II}}^{-\frac{1}{2}} L^2_{\mathrm{b}}(M_{\mathrm{mr}}) \tag{3.47}$$

with the derivatives taken in the sense of extendible distributions. Indeed, choosing a cutoff $\mu \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ which is equal to 1 near 0 the sequence of multiplication operators $1 - \mu(n\rho_{\mathrm{II}})$ tends strongly to the identity on $\rho_{\mathrm{II}}^{-\frac{1}{2}} L^2_{\mathrm{b}}(M_{\mathrm{mr}})$. By assumption this commutes with the elements of $\mathcal{W}$ and it follows that elements with support in the interior of M_{mr}, where ψ_{mr} is a fibration, are dense in D; for these approximation by smooth elements is standard.

That $\Delta+2 : D \longrightarrow D' \subset \mathcal{C}^{-\infty}(M_{\mathrm{mr}})$ is the explicit form of the Riesz representation theorem in this setting. Then the identification, (3.45), of elements of D' follows from the form of Δ. Away from B_{II}, D is a $\mathcal{C}^\infty$ module (since the elements of $\mathcal{W}$ are smooth there) and then (3.45) is the identification of the fiber H^{-1} space. Near B_{II} we may use the explicit form of the Laplacian for the plumbing metric.

Indeed, the local version of the Dirichlet form is

$$D(\phi, \psi) = \int \left(V_{\mathrm{Re}}\phi \overline{V_{\mathrm{Re}}\phi} + V_{\mathrm{Im}}\phi \overline{V_{\mathrm{Im}}\phi} \right) \frac{ds_w d\theta_w}{s_w^2} \tag{3.48}$$

where V is given by (3.28) and it follows that the Laplacian acting on functions on the fibers can be written

$$\Delta = -\frac{\sin^2(\frac{\pi}{1+\rho_z})}{\pi^2 s_t^2} \left(V_{\mathbb{R}}^2 + (\partial_{\theta_z} - \partial_{\theta_w})^2 \right) \tag{3.49}$$

in the coordinates s_w, ρ_z, θ_w and θ_z.

The vector fields $V_{\mathbb{R}}$ and $\rho_{\mathrm{II}}^{-1}(\partial_{\theta_z} - \partial_{\theta_w})$ generate $\mathcal{W}$ near B_{II} over the functions which are constant in θ_w and θ_z. If we write $\mathrm{Diff}^k_{\mathcal{W}}(M_{\mathrm{mr}})$ for the differential operators which can be written as sums of products of elements of at most k elements of $\mathcal{W}$ with smooth coefficients which are independent of the angular variables near B_{II} then

$$\Delta \in \mathrm{Diff}^2_{\mathcal{W}}(M_{\mathrm{mr}}). \tag{3.50}$$

Moreover

$$\begin{aligned} \mathrm{Diff}^1_{\mathcal{W}}(M_{\mathrm{mr}}) : D \longrightarrow \rho_{\mathrm{II}}^{-\frac{1}{2}} L^2_{\mathrm{b}}(M_{\mathrm{mr}}) \text{ and} \\ \mathrm{Diff}^1_{\mathcal{W}}(M_{\mathrm{mr}}) : \rho_{\mathrm{II}}^{-\frac{1}{2}} L^2_{\mathrm{b}}(M_{\mathrm{mr}}) \longrightarrow D' \end{aligned} \tag{3.51}$$

where the second statement follows by duality from the first. Together (3.50) and (3.51) imply (3.45). □

Consider the space $\mathcal{U} \subset \mathcal{V}_{\mathrm{b}}(M_{\mathrm{mr}})$, defined analogously to $\mathcal{W}$, as consisting of the vector fields which commute with ∂_{θ_z} and ∂_{θ_w} near B_{II}. Then let $\mathrm{Diff}^k_{\mathcal{U}}(M_{\mathrm{mr}})$ be the part of the enveloping algebra of $\mathcal{U}$ up to order k, this just consists of the elements of $\mathrm{Diff}^k_{\mathrm{b}}(M_{\mathrm{mr}})$ which commute with ∂_{θ_z} and ∂_{θ_w} near B_{II}. We may define 'higher order' versions of the spaces D and D' :

$$D_k = \{u \in D; \mathrm{Diff}^k_{\mathcal{U}}(M_{\mathrm{mr}}) \cdot u \subset D\}, \\ D'_k = \{u \in D'; \mathrm{Diff}^k_{\mathcal{U}}(M_{\mathrm{mr}}) \cdot u \subset D'\},\ k \in \mathbb{N}. \tag{3.52}$$

Since $\mathcal{U}$ spans $\mathcal{V}_{\mathrm{b}}(M_{\mathrm{mr}})$ over $\mathcal{C}^\infty(M_{\mathrm{mr}})$ it follows that

$$D_k \subset \rho_{\mathrm{II}}^{-\frac{1}{2}} H^k_{\mathrm{b}}(M_{\mathrm{mr}}) \subset D'_k \ \forall\ k. \tag{3.53}$$

Proposition 3.5. *For any k, $\dot{\mathcal{C}}^\infty(M_{\mathrm{mr}})$ is dense in D_k and D'_k and*

$$\Delta + 2 : D_k \longrightarrow D'_k \tag{3.54}$$

is an isomorphism.

Proof. The density statement follows from the same argument as for D and D'.

Consider the commutator relation which follows directly from the definitions

$$[\mathcal{U}, \mathcal{W}] \subset \mathcal{W} \Longrightarrow [\mathrm{Diff}^k_{\mathcal{U}}(M_{\mathrm{mr}}), \Delta] \subset \mathrm{Diff}^2_{\mathcal{W}}(M_{\mathrm{mr}}) \cdot \mathrm{Diff}^{k-1}_{\mathcal{U}}(M_{\mathrm{mr}}),\ k \in \mathbb{N}. \tag{3.55}$$

To prove (3.54) we need to show that if $u \in D$, $Q \in \mathrm{Diff}^k_{\mathcal{U}}(M_{\mathrm{mr}})$ and $f = (\Delta+2)u \in D'_k$ then $Qu \in D$. Assuming the result for $Q \in \mathrm{Diff}^{k-1}_{\mathcal{U}}(M_{\mathrm{mr}})$ it follows from (3.55) that

$$\varDelta Qu = Q\varDelta u + \sum_p L_p Q_p u \text{ with } L_p \in \mathrm{Diff}^2_{\mathcal{W}}(M_{\mathrm{mr}}),\ Q_p \in \mathrm{Diff}^{k-1}_{\mathcal{U}}(M_{\mathrm{mr}})$$
$$\Longrightarrow \varDelta Qu \in D' \Longrightarrow Qu \in D \quad (3.56)$$

by distributional uniqueness. □

Proof of Proposition 3.3. The boundedness (3.39) follows directly from (3.54) and (3.53). □

3.4 Formal solution of $(\varDelta + 2)u = f$

In the previous section the uniform invertibility of $\varDelta + 2$ for the grafted metric was established. In particular the case $k = \infty$ in (3.39) shows the invertibility on conormal functions. In this section we solve the same equation, $(\varDelta + 2)u = f$ in formal power series with logarithmic terms.

Let $\mathcal{C}^\infty_F(M_{\mathrm{mr}}) \subset \mathcal{C}^\infty(M_{\mathrm{mr}})$ denote the subspace annihilated to infinte order at B_{II} by the angular operators D_{θ_z} and D_{θ_w}.

Lemma 3.4. *The restriction, $\varDelta_I$, of the Laplacian to B_I satisfies*

$$(\varDelta_I + 2)^{-1}\left(\rho_{II}(\log\rho_{II})^k g_k\right) = \rho_{II} \sum_{0\le p\le k+1} (\log\rho_{II})^p u_p,\ u_p \in \mathcal{C}^\infty_F(M_{\mathrm{mr}})\ \forall\ g_k \in \mathcal{C}^\infty_F(M_{\mathrm{mr}}). \quad (3.57)$$

Proof. The fiber metric on B_{I} is a trivial family with respect to the product decomposition $B_{\mathrm{I}} = A \times \mathbb{S}$ where A has the complete metric on the Riemann surface with cusps arising from the 'removal' of the nodal points. The Laplacian is therefore essentially self-adjoint and non-negative, so $\varDelta + 2$ is invertible. Either from the form of a parameterix or by Fourier expansion near the cusps it follows that rapid decay in the non-zero Fourier modes (in both angular variables) is preserved by $(\varDelta_{\mathrm{I}} + 2)^{-1}$. Near the boundary the zero Fourier mode satisfies a reduced, ordinary differential, equation with regular singular points and having indicial roots 1 and -2 in terms of a defining function for the (resolved) cusps. Then (3.57) follows directly. $\square$

Lemma 3.5. *If $u \in \mathcal{C}_F^\infty(M_{\mathrm{mr}})$ then $\Delta u \in \mathcal{C}_F^\infty(M_{\mathrm{mr}})$ restricts to B_{II} to $\widetilde{\Delta}_{II} v$, $v = u\big|_{B_{II}}$ where $\widetilde{\Delta}_{II}$ is an ordinary differential operator of order 2 elliptic in the interior with regular singular endpoints, with indicial roots $-1, 2$ such that*

$$\mathrm{Nul}(\widetilde{\Delta}_{II} + 2) \subset \rho_I^{-1} \mathcal{C}^\infty(B_{II}) \tag{3.58}$$

has no smooth elements and for $h_j \in \mathcal{C}_F^\infty(B_{II})$

$$\begin{gathered}(\widetilde{\Delta}_{II} + 2)^{-1} (\log \rho_I)^j h_j = \sum_{0 \le q \le j} (\log \rho_I)^q v_{q,j} + \rho_I^2 (\log \rho_I)^{j+1} w_j \\ \text{with } v_{q,j},\ w_j \in \mathcal{C}_F^\infty(B_{II}).\end{gathered} \tag{3.59}$$

Proof. The form of the Laplacian in (3.49) shows that the reduced operator $\widetilde{\Delta}_{\mathrm{II}}$ exists and after the change coordinates on B_{II} to

$$\rho = \frac{1}{1 + \rho_{\mathrm{II}}} \tag{3.60}$$

becomes

$$\Delta + 2 = 2 - (\frac{\sin(\pi\rho)}{\pi\rho})^2 [(\rho\partial_\rho)^2 - \rho\partial_\rho]. \tag{3.61}$$

The indicial roots of this operator are 2 and -1 and its homoeneity shows that the null space has no logarithmic terms. The absence of smooth elements in the null space follows by integration by parts and positivity. $\square$

The problem that we need to solve at B_{II} is

$$(\Delta + 2)(\rho_{\mathrm{II}} w) = \rho_{\mathrm{II}} g + O(\rho_{\mathrm{II}}^2) \Longrightarrow (\widetilde{\Delta}_{\mathrm{II}}^{(1)} + 2)(w\big|_{B_{\mathrm{II}}}) = g\big|_{B_{\mathrm{II}}}. \tag{3.62}$$

Since the parameter, s_t, is the product of defining functions for B_{I} and B_{II} and commutes through the problem this can be solved by dividing by it. Thus $\widetilde{\Delta}_{\mathrm{II}}^{(1)}$ is obtained from $\widetilde{\Delta}_{\mathrm{II}}$ by conjugating by a boundary defining function on B_{II} so the preceding Lemma can be applied after noting the shift of the indicial roots.

Lemma 3.6. *For the conjugated operator on B_{II},*

$$\mathrm{Nul}(\widetilde{\Delta}_{II}^{(1)} + 2) \subset \mathcal{C}^\infty(B_{II}) \tag{3.63}$$

with the Dirichlet problem uniquely solvable and

$$\begin{gathered}(\widetilde{\Delta}_{II} + 2)^{-1}(\log \rho_I)^j h_j = \sum_{0 \le q \le j} (\log \rho_I)^q v_{q,j} + \rho_I^3 (\log \rho_I)^{j+1} w_j \\ \text{with } v_{q,j},\ w_j \in \mathcal{C}_F^\infty(B_{II}).\end{gathered} \tag{3.64}$$

To express the form of the expansion which occur below, consider the space of polynomials in $\log \rho_{\mathrm{I}}$ and $\log \rho_{\mathrm{II}}$ with coefficients in $\mathcal{C}_F^\infty(M_{\mathrm{mr}})$

$$\mathcal{P}^k = \left\{ u = \sum_{0 \le l+p \le k} (\log \rho_{\mathrm{I}})^l (\log \rho_{\mathrm{II}})^p u_{l,p},\ u_{l,p} \in \mathcal{C}_F^\infty(M_{\mathrm{mr}}) \right\}. \tag{3.65}$$

We also consider the filtration of these spaces by the maximal order in each of the variables:

$$\begin{aligned}\mathcal{P}_{\mathrm{I}}^{k,j} &= \left\{ u = \sum_{0 \le l+p \le k,\ l \le j} (\log \rho_{\mathrm{I}})^l (\log \rho_{\mathrm{II}})^p u_{l,p},\ u_{l,p} \in \mathcal{C}_F^\infty(M_{\mathrm{mr}}) \right\},\ j \le k \\ \mathcal{P}_{\mathrm{II}}^{k,m} &= \left\{ u = \sum_{0 \le l+p \le k,\ p \le m} (\log \rho_{\mathrm{I}})^l (\log \rho_{\mathrm{II}})^p u_{l,p},\ u_{l,p} \in \mathcal{C}_F^\infty(M_{\mathrm{mr}}) \right\},\ m \le k.\end{aligned} \tag{3.66}$$

Since the coefficients are in $\mathcal{C}_F^\infty(M_{\mathrm{mr}})$, Δ acts as a smooth b-differential operator on all of these spaces. If $u \in \mathcal{P}_{\mathrm{I}}^{k,p}$, then $u = u_p + u'$ with $u' \in \mathcal{P}_{\mathrm{I}}^{k,p-1}$ and $u_p = v(\log \rho_{\mathrm{I}})^p$ where $v \in \mathcal{P}_{\mathrm{I}}^{k-p,0}$. Then $\Delta u = (\Delta_{\mathrm{I}} v)(\log \rho_{\mathrm{I}})^p + f'$, $f' \in \mathcal{P}_{\mathrm{I}}^{k-1,p-1} + \rho_{\mathrm{I}} \mathcal{P}_{\mathrm{I}}^{k,p-1}$ where the first error term corresponds to at least one derivation of $(\log \rho_{\mathrm{I}})^p$. Similar statements apply to B_{II} and $\tilde{\Delta}_{\mathrm{II}}$.

As a basis for iteration, to capture the somewhat complicated behavior of the logarthimic terms, we first consider a partial result.

Proposition 3.6. *For each k*

$$f \in \rho_{II}\mathcal{P}^k + \rho_I\rho_{II}\mathcal{P}^{k+1} \Longrightarrow \exists\ u \in \rho_{II}\mathcal{P}^{k+1} + \rho_I^2\rho_{II}\mathcal{P}_{II}^{k+2,k+1} \tag{3.67}$$

such that

$$(\varDelta + 2)u - f \in s_t\left(\rho_{II}\mathcal{P}^{k+1} + \rho_I\rho_{II}\mathcal{P}^{k+2}\right). \tag{3.68}$$

Proof. We first solve on B_{I}, then on B_{II}. The second term in f in (3.67) vanishes on B_{I} so the restriction $f_{\mathrm{I}} \in \rho_{\mathrm{II}}\mathcal{P}^k\big|_{B_{\mathrm{I}}}$. Proceeding iteratively, suppose

$$f \in \rho_{\mathrm{II}}\mathcal{P}_{\mathrm{I}}^{k,j} + \rho_{\mathrm{I}}\rho_{\mathrm{II}}\mathcal{P}^{k+1}$$

with $j \le k$ and consider the term of order j in $\log\rho_{\mathrm{I}}$; this is a polynomial in $\log\rho_{\mathrm{II}}$ of degree at most $k - j$ with coefficients in $\rho_{\mathrm{II}}\mathcal{C}_F^\infty(B_{\mathrm{I}})$. Applying Lemma 3.4 to the restriction to B_{I} gives a polynomial in $\log\rho_{\mathrm{II}}$ of degree at most $k - j + 1$ with coefficients in $\rho_{\mathrm{II}}\mathcal{C}_F^\infty(B_{\mathrm{I}})$. Extending these coefficients off B_{I} and restoring the coefficient of $(\log\rho_{\mathrm{I}})^j$ gives $v_j \in \rho_{\mathrm{II}}\mathcal{P}_{\mathrm{I}}^{k+1,j}$ such that

$$(\varDelta + 2)v_j - f = -f',\ \ f' \in \rho_{\mathrm{II}}\mathcal{P}_{\mathrm{I}}^{k,j-1} + \rho_{\mathrm{I}}\rho_{\mathrm{II}}\mathcal{P}^{k+1}.$$

Here the first part of the error arises from differentiation of the factor $(\log\rho_{\mathrm{I}})^j$ in v_j at least once. If we start with $j = k$ and proceed iteratively over decreasing j this allows us to find $v \in \rho_{\mathrm{II}}\mathcal{P}^{k+1}$ such that

$$(\varDelta + 2)v - f = -g \in \rho_{\mathrm{I}}\rho_{\mathrm{II}}\mathcal{P}^{k+1}. \tag{3.69}$$

Now we proceed similarly by solving on B_{II} using Lemma 3.6. So, suppose $h \in \rho_{\mathrm{I}}\rho_{\mathrm{II}}\mathcal{P}_{\mathrm{II}}^{k+1,p}$, for $p \le k+1$. Then the coefficient h_p of $(\log\rho_{\mathrm{II}})^p$ is a polynomial of degee at most $k + 1 - p$ in $\log\rho_{\mathrm{I}}$ with coefficients in $\rho_{\mathrm{I}}\rho_{\mathrm{II}}\mathcal{C}_F^\infty(M_{\mathrm{mr}})$. Conjugating away the factor of ρ_{II} and applying Lemma 3.6 to the restriction to B_{II} and then extending the coefficients off B_{II} allows us to find $w_p \in \rho_{\mathrm{I}}\rho_{\mathrm{II}}\mathcal{P}_{\mathrm{II}}^{k+1,p} + \rho_{\mathrm{I}}^2\rho_{\mathrm{II}}\mathcal{P}_{\mathrm{II}}^{k+2,p}$, where the second term arises from the possible increase in multiplicity of the logarithmic coefficient of

ρ_{I}^2 in the solution, satisfying

$$(\Delta+2)w_p - g = -g' + e,\ g' \in \rho_{\mathrm{I}}\rho_{\mathrm{II}}\mathcal{P}_{\mathrm{II}}^{k+1,p-1},\ e \in \rho_{\mathrm{I}}\rho_{\mathrm{II}}^2\mathcal{P}_{\mathrm{II}}^{k+1,p} + \rho_{\mathrm{I}}^2\rho_{\mathrm{II}}^2\mathcal{P}_{\mathrm{II}}^{k+2,p} \quad (3.70)$$

where the first part of the error arises from differentiation of $(\log\rho_{\mathrm{II}})^p$ at least once. Starting with $p = k+1$ and iterating over decreasing p allows us to find $w \in \rho_{\mathrm{I}}\rho_{\mathrm{II}}\mathcal{P}^{k+1} + \rho_{\mathrm{I}}^2\rho_{\mathrm{II}}\mathcal{P}^{k+2,k+1}$ such that

$$(\Delta+2)w - g \in \rho_{\mathrm{I}}\rho_{\mathrm{II}}^2\mathcal{P}^{k+1} + \rho_{\mathrm{I}}^2\rho_{\mathrm{II}}^2\mathcal{P}^{k+2}. \quad (3.71)$$

Combining (3.69) and (3.71) gives (3.68) since $\rho_{\mathrm{I}}\rho_{\mathrm{II}}$ is a smooth multiple of s_t. □

Proposition 3.6 allows iteration since s_t commutes through $\Delta + 2$.

Proposition 3.7. *If $f \in \rho_{II}\mathcal{P}^k + \rho_I\rho_{II}\mathcal{P}^{k+1}$ then $u = (\Delta+2)^{-1}f \in s_t^{-\epsilon}H_{\mathrm{b}}^{\infty}(M_{\mathrm{mr}})$ for any $\epsilon > 0$, has a complete asymptotic expansion of the form*

$$u \simeq \sum_{j\geq 0} s_t^j u_j,\ u_j \in \rho_{II}\mathcal{P}^{k+j} + \rho_I\rho_{II}\mathcal{P}_{II}^{k+j+1,k+j}. \quad (3.72)$$

Proof. For any $\epsilon > 0$, $g = s_t^{\epsilon} f \in \rho_{\mathrm{II}}^{-\frac{1}{2}}H_{\mathrm{b}}^{\infty}(M_{\mathrm{mr}})$ so $u = s_t^{-\epsilon}(\Delta+2)^{-1}g$ exists by (3.39). Comparing u to the expansion cut off at a finite point gives (3.72). □

This result can itself be iterated, asymototically summed and then the rapidly decaying remainder term again removed to show the polyhomogeneity of the solution for an asymptotically covergent sum over terms on the right in (3.72).

For the solution of the curvature equation the leading term is smooth because of the special structure of the forcing term.

Lemma 3.7. *If $f \in \mathcal{C}^{\infty}(M_{\mathrm{mr}})$ has support disjoint from B_{II} then $u = (\Delta+2)^{-1}f$ is log-smooth and has an asymptotic expansion of the form*

$$u \simeq \rho_{II}v_0 + \sum_{k\geq 1} s_t^k v_k,\ v_k \in \rho_{II}\mathcal{P}^k + \rho_I\rho_{II}\mathcal{P}_{II}^{k+1,k}. \quad (3.73)$$

Note that log-smoothness follows from the fact that $s_t = a\rho_{\mathrm{I}}\rho_{\mathrm{II}}$, $a \in \mathcal{C}^\infty_F(M_{\mathrm{mr}})$ so each term in the expansion can be written as a polynomial in ρ_{I}, $\rho_{\mathrm{I}} \log \rho_{\mathrm{I}}$, ρ_{II} and $\rho_{\mathrm{II}} \log \rho_{\mathrm{II}}$ of degree at least $2k$.

3.5 Polyhomogeneity for the curvature equation

Under a conformal change from the grafted metric h with curvature R to $e^{2f}h$ the condition for the curvature of the new metric to be -1 given by (3.9). To construct the canonical metrics on the fibers we proceed, as in the linear case discussed above, to solve (3.9) in the sense of formal power series at the two boundaries above $s_t = 0$ and then, using the Implicit Function Theorem deduce that the actual solution has this asymptotic expansion.

Lemma 3.8. *For the grafted metric there is a formal power series*

$$\sum_{k\geq 2} s_t^k f_k, \;\; f_2 \in \mathcal{C}^\infty_F(M_{\mathrm{mr}}), \;\; f_k \in \rho_{II}\mathcal{P}^{k-2} + \rho_I\rho_{II}\mathcal{P}^{k-1,k-2}_{II}, \;\; k \geq 3, \tag{3.74}$$

solving (3.9).

The $\mathcal{P}^k$ are defined in (3.65); in the last term there is no factor of $(\log \rho_{\mathrm{II}})^{k-1}$.

Proof. Since $R + 1 \in s_t^2\mathcal{C}^\infty(M_{\mathrm{mr}})$ is supported away from B_{II}, Lemma 3.7 shows that $g_1 = -(\varDelta + 2)^{-1}(R + 1)$ is of the form (3.74). We look for the formal power series solution of the non-linear problem as

$$f \simeq \sum_{k\geq 1} g_k \tag{3.75}$$

Inserting this sum into the equation gives

$$-(\varDelta + 2)(\sum_{i\geq 1} g_i) = \sum_{j\geq 2} \frac{2^j}{j!}(g_1 + \sum_{k\geq 2} g_k)^j + 1 + R. \tag{3.76}$$

For each $i \geq 2$ we fix g_i by

$$-(\Delta+2)g_i = \sum_{j\geq 1} \frac{2^j}{j!}(g_1 + \sum_{i-1\geq k\geq 2} g_k)^j - \frac{2^j}{j!}(g_1 + \sum_{i-2\geq k\geq 2} g_k)^j$$
$$= g_{i-1}P_i(g_1, g_2, ...g_{i-1}) \quad (3.77)$$

where P_i is a formal power series in $g_1, ...g_{i-1}$ without constant term.

Proceeding by induction we claim that

$$g_i \simeq \sum_{j\geq 2i} s_t^j g_{i,j},\ \ g_{i,j} \in \rho_{\mathrm{II}}\mathcal{P}^{j-2i} + \rho_{\mathrm{I}}\rho_{\mathrm{II}}\mathcal{P}_{\mathrm{II}}^{j-2i+1,j-2i}. \quad (3.78)$$

We have already seen that this holds for $i=1$ and using the obvious multiplicitivity properties

$$\mathcal{P}^k \cdot \mathcal{P}^j \subset \mathcal{P}^{j+k},\ \ \mathcal{P}^k \cdot \mathcal{P}_{\mathrm{II}}^{j,j-1} \subset \mathcal{P}_{\mathrm{II}}^{j+k,j+k-1}$$

it follows from the inductive assumption, that (3.78) holds for all smaller indices, that

$$g_{i-1}P_i(g_1, g_2, ...g_{i-1})$$
$$\simeq s_t^{2i} \sum_{k\geq 2, j\geq 2i-2} \left(\rho_{\mathrm{II}}\mathcal{P}^{j-2i+2} + \rho_{\mathrm{I}}\rho_{\mathrm{II}}\mathcal{P}_{\mathrm{II}}^{j-2i+3,j-2i+2}\right)\left(\rho_{\mathrm{II}}\mathcal{P}^{k-2} + \rho_{\mathrm{I}}\rho_{\mathrm{II}}\mathcal{P}_{\mathrm{II}}^{k-1,k-2}\right) \quad (3.79)$$
$$\simeq \sum_{k\geq 2i} s_k^j F_k,\ \ F_k \in \rho_{\mathrm{II}}\mathcal{P}^{k-2i} + \rho_{\mathrm{I}}\rho_{\mathrm{II}}\mathcal{P}_{\mathrm{II}}^{k-2i+1,k-2i}.$$

Applying Proposition 3.7 we recover the inductive hypothesis at the next step. Then (3.74) follows from (3.75) and (3.78). □

Summing the formal power series solution gives a polyhomogeneous function with

$$-\Delta f_0 = R + e^{2f_0} + g, g \in O(s_t^\infty). \quad (3.80)$$

Now we look for the solution as a perturbation $f = f_0 + \tilde{f}$, so $\tilde{f}$ satisfies

$$-\Delta \tilde{f} = -g + e^{2f_0}(e^{2\tilde{f}} - 1). \quad (3.81)$$

which can be rewritten as

$$\tilde{f} = -(\Delta + 2)^{-1}\left(2\tilde{f}(e^{2f_0} - 1) + e^{2f_0}(e^{2\tilde{f}} - 1 - 2\tilde{f}) - g\right).$$

So consider the nonlinear operator

$$K : \tilde{f} \mapsto (\Delta + 2)^{-1}\left(2\tilde{f}(e^{2f_0} - 1) + e^{2f_0}(e^{2\tilde{f}} - 1 - 2\tilde{f}) - g\right) \tag{3.82}$$

which acts on $s_t^N H_b^M(M_{\mathrm{mr}})$ for all $N \geq 1$ and $M > 2$. Note that for $M > 2$, the b-space $H_b^M(M_{\mathrm{mr}})$ is closed under multiplication, therefore this weighted Sobolev space is also an algebra. Since the nonlinear terms are at least quadratic, K is well-defined on this domain. The solution to (3.81) satisfies $\tilde{f} = K(\tilde{f})$.

Proposition 3.8. *For any $M > 1$ and $N \geq 1$ there is a unique solution $\tilde{f} \in s_t^N H_b^M(M_{\mathrm{mr}})$ to the equation* (3.81).

Proof. We construct the solution $\tilde{f}$ by iteration. Let $\tilde{f} = s_t^N \sum_{i\geq 2} {s_t}^i f_i$, put it into equation (3.81), divide by the common factor s_t^N on both sides and then we get

$$\sum_{i\geq 2} s_t^i f_i = K(\sum s_t^i f_i) = (\Delta + 2)^{-1}\left((e^{2f_0} - 1)\sum s_t^i f_i + s_t^N(\sum s_t^i f_i)^2 + s_t^{-N} g\right) \tag{3.83}$$

The right hand side belongs to $(\Delta+2)^{-1}(O(s_t^2))$ because of the quadratic structure and the fact that $e^{2f_0} - 1 \in O(s_t^2)$. Therefore the right hand side is the form $(\Delta+2)^{-1}(s_t h)$ where $s_t h \in \rho_{\mathrm{II}}^{-\frac{1}{2}} H_{\mathrm{b}}^M(M_{\mathrm{mr}})$ so this quantity is well-defined using Proposition 3.3.

Now we proceed by induction. Assume that the first k terms in the expansion have been solved, then the equation for the next term f_k is given by

$$f_k = (\Delta + 2)^{-1}\left((e^{2f_0} - 1)f_{k-2} + s_t^N Q(f_0, \ldots f_{k-1})\right).$$

where the polynomial Q on the right hand side is a quadratic polynomial of order $k - N$. By using the invertibility property in Proposition 3.3, we can now solve f_k. Therefore the induction gives us the total expansion for $\tilde{f}$. □

Proof of Theorem 3.1. From Proposition 3.8 we obtain the solution, $f = f_0 + \tilde{f}$, to the curvature equation $R(e^{2f}h) = -1$. Since f_0 is the formal power series and $\tilde{f} \in s_t^\infty \mathcal{C}^\infty(M_{\mathrm{mr}})$, we get the solution with required regularity. □

Chapter 4

Group resolution

4.1 Introduction

For a general compact Lie group G acting on a smooth compact manifold with corners M, Albin and Melrose [1] showed that there is a canonical full resolution such that the group action lifts to the blow-up space $Y(M)$ to have a unique isotropy type. This generalized the result of Borel [4] that if all the isotropy groups of a compact group action are conjugate then the orbit space $G\backslash M$ is smooth.

In this paper, we give an explicit construction of such a resolution of the unitary group action on the space of self-adjoint matrices

$$S = S(n) = \{X \in M_n(\mathbb{C}) | X^* = X\}$$

with the unitary group $U(n)$ acting by conjugation: for $u \in U(n), X \in S$,

$$u \cdot X := uXu^{-1}.$$

The orbit of an element $X \in S$, denoted by $U(n) \cdot X$, consists of the matrices with the same eigenvalues including multiplicities. For a matrix $X \in S$ with m distinct eigenvalues $\{\lambda_j\}_{j=1}^m$, each with multiplicity $i_k, k = 1, 2, .., m$, the isotropy group of X

is isomorphic to a direct sum of smaller unitary groups:

$$U(n)^X = \{u \in U(n) \mid u \cdot X = X\} \cong \oplus_{k=1}^m U(i_k).$$

Thus the matrices with the same multiplicities $\{i_k\}$, have conjugate isotropy groups. The isotropy types are therefore parametrized by the partition of n into integers. Note here that the partition contains information about ordering of the eigenvalues, for example, the two partitions of 3, $\{i_1 = 1, i_2 = 2\}$ and $\{i_1 = 2, i_2 = 1\}$, are not the same type.

For $n > 1$, the eigenvalues are not smooth functions on S, but are singular where the multiplicities change. We will show that, by doing an iterative blow up, the singularities are resolved and the eigenvalues become smooth functions on the resolved space.

Recall the lemma of group action resolution in [1]:

Lemma 4.1 ([1]). *A compact manifold (with corners), M, with a smooth, boundary intersection free, action by a compact Lie group, G, has a canonical full resolution, $Y(M)$, obtained by iterative blow-up of minimal isotropy types.*

Consider the trivial bundle over S,

$$M := S \times \mathbb{C}^n,$$

the fiber of which can be decomposed into n eigenspaces of the self-adjoint matrix at the base point. This decomposition is not unique at matrices with multiple eigenvalues and in general the eigenspaces are not smooth.

There are two basic kinds of real blow up, namely radial and projective, which give different results; radial blow up of a hypersurface produces a new boundary while projective blow up does not. As pointed out in [1], projective blow up usually requires an extra step of reflection in the iterative scheme in order to obtain smoothness. We will show that, after radial blow up, the trivial bundle M decomposes into the direct sum of n 1-dimensional eigenspaces. In contrast, after the projective blow up, though the eigenvalues are still smooth on the resolved space and locally this is a smooth decomposition into simple eigenspaces, but the trivial bundle doesn't split into global line bundles.

Next we recall the resolution in the sense of Albin and Melrose.

Definition 4.1 (eigenresolution). *By an eigenresolution of S, we mean a manifold with corners $\hat{S}$, with a surjective smooth map $\beta : \hat{S} \to S$ such that the self-adjoint matrices have a smooth (local) diagonalization when lifted to $\hat{S}$, with eigenvalues lifted to smooth functions on $\hat{S}$.*

Note in the definition we only require the the diagonalization exists locally. To encompass the information of global decomposition of eigenvectors, we introduce the full resolution below.

Definition 4.2 (full eigenresolution). *A full eigenresolution is an eigenresolution with global eigenbundles. The eigenvalues are lifted to n smooth functions f_i on $\hat{S}$, and M, which is the trivial n-dimensional complex vector bundle on $\hat{S}$, is decomposed into n smooth line bundles:*

$$\hat{S} \times \mathbb{C}^n = \bigoplus_{i=1}^{n} E_i$$

such that

$$\beta(x)v_i = f_i(x)v_i, \forall\ v_i \in E_i(x), \forall\ x \in \hat{S}.$$

We use the blow-up constructions introduced by Melrose in the book [34, Chapter 5] and show that we can obtain resolutions in this way and, in particular, full resolution if we use radial blow-up.

Theorem 4.1. *The iterative blow up of the isotropy types in S, in an order compatible with inclusion of the conjugation class of the isotropy group, yields an eigenresolution. In particular, radial blow up gives a full eigenresolution.*

4.2 Proof of the theorem

The proof proceeds through induction on dimension. We begin the proof by discussing the first example which is the 2×2 matrices.

Lemma 4.2 (2 × 2 case)**.** *For the* 2×2 *matrices* $S(2)$*, the eigenvalues and eigenvectors are smooth except at the scalar matrices. After radial blow up, the singularities are resolved and the trivial 2-dim bundle splits into the direct sum of two line bundles. The projective blow up also gives smooth eigenvalues, but does not give two global line bundles..*

Proof. In this case $S = S(2) = \left\{ \left(\begin{array}{cc} a_{11} & z_{12} \\ \bar{z}_{12} & a_{22} \end{array} \right) \middle| a_{ii} \in \mathbb{R}, z_{12} \in \mathbb{C} \right\} \cong \mathbb{R}^4$. Thus S is isomorphic to the product of $\mathbb{R}$ and the trace-free subspace

$$S_0 = \left\{ \left(\begin{array}{cc} a_{11} & z_{12} \\ \bar{z}_{12} & a_{22} \end{array} \right) \middle| a_{11} + a_{22} = 0 \right\}, \tag{4.1}$$

i.e. there is a bijective linear map:

$$\begin{aligned} \phi: \quad & S \to S_0 \times \mathbb{R} \\ A = & \left(\begin{array}{cc} a_{11} & z_{12} \\ \bar{z}_{12} & a_{22} \end{array} \right) \mapsto (A_0 := A - (a_{11} + a_{22})I, a_{11} + a_{22}) \end{aligned} \tag{4.2}$$

The eigenvalues λ_i and eigenvectors v_i of A are related to those A_0 by $\lambda_i(A) = \lambda_i(A_0) + tr(A)$, $v_i(A) = v_i(A_0), i = 1, 2$. Therefore, we can restrict the discussion of resolution to the subspace S_0, since the smoothness of eigenvalues and eigenvectors on S follows.

Let $z_{12} = c + di$. The space S_0 can be identified with $\mathbb{R}^3 = \{(a_{11}, c, d)\}$. The eigenvalues of this matrix are:

$$\lambda_{\pm} = \pm \sqrt{a_{11}^2 + c^2 + d^2}. \tag{4.3}$$

Hence the only singularity of the eigenvalues on S_0 is at the point $a_{11} = c = d = 0$ which is the zero matrix.

Based on the resolution formula in [34], the radial blow up can be realized as

$$\hat{S}_0 = [S_0, \{0\}] = S^+N\{0\} \sqcup (S_0 \setminus \{0\}) \simeq \mathbb{S}^2 \times [0, \infty)_+ \tag{4.4}$$

where the front face $S^+N\{0\} \simeq \mathbb{S}^2$. Here the radial variable r is $\sqrt{a_{11}^2 + c^2 + d^2}$. The blow-down map is

$$\beta : [S_0, \{0\}] \to S_0, (r, \theta) \mapsto r\theta, r \in \mathbb{R}_+, \theta \in \mathbb{S}^2. \tag{4.5}$$

The radial variable r lifts to be a smooth on the blown up space, therefore the two eigenvalues $\lambda_\pm = \pm r$ become smooth functions.

Now we consider the eigenvectors of the corresponding eigenvalues $\lambda_\pm$

$$v_\pm = (c + di, \pm\sqrt{a_{11}^2 + c^2 + d^2} - a_{11}) \in \mathbb{C}^2. \tag{4.6}$$

Similar to the discussion of the eigenvalues, the only singularity is at $r = 0$, which becomes a smooth function on $[S_0, \{0\}]$, it follows that v_+ and v_- span two smooth line bundles on $[S_0, \{0\}]$.

If we do the projective blow up instead, which identifies the antipodal points in the front face of $\mathbb{S}^2$ to get $\mathbb{RP}^2$, namely

$$\tilde{S}_0 = \{(x, l) | x \in l\} \subset \mathbb{R}^3 \times \mathbb{RP}^2 \tag{4.7}$$

for which we will cover it with three coordinate patches

$$(x_1, y_1, z_1) = (c, \frac{d}{c}, \frac{a_{11}}{c}) \in \mathbb{R}^3$$

and the other two $(x_2, y_2, z_2), (x_3, y_3, z_3)$=$(d, \frac{c}{d}, \frac{a_{11}}{d}), (a_{11}, \frac{c}{a_{11}}, \frac{d}{a_{11}})$ are similar. The two eigenvalues we get from here are

$$v_\pm = \pm\sqrt{a_{11}^2 + c^2 + d^2} = \pm|x_1|\sqrt{(1 + y_1^2 + z_1^2)}.$$

which is smooth at $\{x_1 > 0\}$. Similar discussions hold for the other two coordinate patches.

However, the trivial bundle does not decompose into two line bundles as in the radial case. The nontriviality of eigenbundles can be seen by taking a loop in $\mathbb{RP}^2$

$$l = \beta^{-1}(\{r = 1\}) \subset \tilde{S}$$

which is a curve that winds twice around origin. This curve intersects the line $c = d = 0$ twice, which hits at two different places thus both $a_{11}^{\pm} = \pm 1$ are on the curve, and (4.6) shows that starting from $v_- = (0, -2) = (0, -2a_{11}^{+})$, this turns into $v_+ = (0, -2) = (0, 2a_{11}^{-})$, which means they are not separated by projective blow up.

Now that we have done the radial resolution for the trace free slice S_0, the resolution of S follows. Consider S as a 3-dim vector bundle on $\mathbb{R}$ with trace being the projection map, then at each base point λ, the fiber is $S_0 + \lambda I$. The resolution is $[S_0 + \lambda I; \lambda I] \cong [S_0; 0]$. Since the trace direction is transversal to the blow up, and therefore

$$[S; \mathbb{R}I] = [S_0; \{0\}] \times \mathbb{R}. \tag{4.8}$$

And because the trace don't change the eigenvectors, the smoothness follows. □

To proceed to higher dimensions, we first discuss the partition of eigenvalues into clusters. The basic case is when the eigenvalues are divided into two clusters, then the $U(n)$ action of the matrices can be decomposed to two commuting actions.

Definition 4.3 (spectral gap)**.** *A connected neighborhood $U \subset S$ has a spectral gap at $c \in \mathbb{R}$, if c is not an eigenvalue of X, for any $X \in U$.*

Note here that since U is connected, the number of eigenvalues less than c stays the same for all $X \in U$, denoted by k.

Lemma 4.3 (local eigenspace decomposition)**.** *If a neighborhood $U \subset S(n)$ has a spectral gap at c, then the matrices in U can be decomposed into two self-adjoint commuting matrices smoothly:*

$$X = L_X + R_X, L_X R_X = R_X L_X.$$

with $\mathrm{rank}(L_X) = k$*,* $\mathrm{rank}(R_X) = n - k$*.*

Proof. Let γ be a simple closed curve on $\mathbb{C}$ such that it intersects with $\mathbb{R}$ only at -R and c, where R is a sufficiently large number. In this way, for any matrix $X \in U$, the k smallest eigenvalues are all contained inside γ. We consider the operator

$$P_X : \mathbb{C}^n \to \mathbb{C}^n$$

$$P_X := -\frac{1}{2\pi i} \oint_\gamma (X - sI)^{-1} ds \tag{4.9}$$

Since the resolvent is nonsingular on γ, P_X is a well-defined operator and varies smoothly with X. And the integral is independent of choice of γ up to homotopy.

First we show that P_X is a projection operator, i.e.

$$P_X^2 = P_X$$

Let γ_s and γ_t be two curves satisfying the above condition with γ_s completely inside γ_t, then

$$\begin{aligned} P_X^2 &= -\tfrac{1}{4\pi^2} \oint_{\gamma_t} (X - tI)^{-1} dt (\oint_{\gamma_s} (X - sI)^{-1} ds) \\ &= -\tfrac{1}{4\pi^2} \oint_{\gamma_t} dt [\oint_{\gamma_s} \tfrac{1}{s-t} (X - sI)^{-1} ds - \oint_{\gamma_s} \tfrac{1}{s-t} (X - tI)^{-1} ds] \\ &= I - II \end{aligned}$$

where using the fact that s is completely inside γ_t

$$I = -\frac{1}{4\pi^2} \oint_{\gamma_s} \frac{1}{X - sI} ds \oint_{\gamma_t} \frac{1}{s-t} dt = -\frac{1}{4\pi^2} (-2\pi i) \oint_{\gamma_s} \frac{1}{X - sI} ds = P_X$$

and any t on γ_t is outside of the loop γ_s

$$\oint_{\gamma_s} \frac{1}{s-t} ds = 0$$

we have

$$II = -\frac{1}{4\pi^2} \oint_{\gamma_t} (X - tI)^{-1} dt \oint_{\gamma_s} \frac{1}{s-t} ds = 0$$

Therefore $P_X^2 = P_X$.

Then we show that P_X is self-adjoint. This is because

$$P_X^* = \frac{1}{2\pi i}\int_\gamma ((X-sI)^{-1})^* d\bar{s} = \frac{1}{2\pi i}\int_{-\bar{\gamma}} (X-sI)ds = P_X.$$

P_X maps $\mathbb{R}^n$ to the invariant subspace spanned by the eigenvectors corresponding to eigenvalues that are less than c. We denote this invariant subspace by L and its orthogonal complement by R. Write X as the diagonalization $X = V\Lambda V^{-1}$ where Λ is the eigenvalue matrix and V consists its eigenvectors as columns. Then L is spanned by the first k columns of V. Take one of the eigenvectors $v_j \in L, j = 1, 2, ..., k$,

$$\begin{aligned} P_X v_j &= -\frac{1}{2\pi i}\oint_\gamma (X-sI)^{-1} v_j ds \\ &= -\frac{1}{2\pi i}\oint V(\Lambda - sI)^{-1}V^{-1}v_j \\ &= -\frac{1}{2\pi i} v_j \oint \frac{1}{\lambda_j - s} ds = v_j. \end{aligned}$$

Similarly for $v_j \in R$ that corresponds to an eigenvalue greater than c (therefore λ_j is outside the loop),

$$P_X v_j = -\frac{1}{2\pi i} v_j \oint \frac{1}{\lambda_j - s} ds = 0,$$

therefore

$$(I - P_X)v_j = v_j, \forall v_j \in R.$$

Then using the projection P_X we define two operators L_X and R_X as

$$L_X := P_X X P_X \tag{4.10}$$

and

$$R_X := (I - P_X)X(I - P_X). \tag{4.11}$$

Since P_X is smooth, the two operators are also smooth. Moreover, using the fact that P_X is a projection onto the invariant subspace L, we have

$$(I - P_X)XP_X = P_X X(I - P_X) = 0,$$

therefore

$$X = L_X + R_X.$$

For an eigenvector $v \in L$,

$$L_X v = Xv, \; R_X v = 0, \tag{4.12}$$

i.e. L_X equals to X when restricted to L, similarly $R_X|_R = X$. Since $P_X^* = P_X$, L_X and R_X are also self-adjoint. In this way we get two commuting lower rank matrices L_X and R_X. □

It is natural to to have a finer decomposition when there is more than one spectral gap in the neighborhood, and we have the following corollary.

Corollary 2. *If the eigenvalues of matrices in a neighborhood U can be grouped into k clusters, Then the matrices can be decomposed into k lower rank self-adjoint commuting matrices smoothly.*

Proof. Do the decomposition inductively. If k=2, then it is the case in Lemma 4.3. Suppose the decomposition for $k = l - 1$ is defined. Then for $k = l$, since the eigenvalues can also be divided into 2 clusters (by combining the smallest l-1 groups of eigenvalues together), then $X = L_X + R_X$, with L_X and R_X corresponding to the two intervals. Then L_X satisfies the separation condition for $l - 1$ clusters, so by induction, $L_X = L_1 + ... + L_{l-1}$. Therefore, $X = L_1 + L_2 + ... + L_{l-1} + R_X$ is the desired division. □

Using the above Lemma 4.3 of decomposition of matrices in a neighborhood, we can now show that locally the trivial bundle $S \times \mathbb{C}^n$ decomposes into two subspaces if there is a spectral gap. And moreover, locally there is a product structure of two lower order matrices. In order to see this, we need to introduce the Grassmannian. Let $Gr_{\mathbb{C}}(n, k)$ denote the Grassmannian, i.e. the set of k-dim subspace in $\mathbb{C}^n$. Consider the tautological vector bundle over Grassmannian:

$$\pi : T_k \to Gr_{\mathbb{C}}(n, k), \pi^{-1}(p) = V(p).$$

where each fibre is a k-dimensional subspace in $\mathbb{C}^n$, with self-adjoint operators acting on it. Similarly, we define T_{n-k} to be the orthogonal complement of T_k:

$$\pi : T_{n-k} \to Gr_{\mathbb{C}}(n,k), \pi^{-1}(p) = V(p)^{\perp}.$$

Definition 4.4 (operator bundle)**.** *Let P_k (resp. P_{n-k}) be the bundles over $Gr_{\mathbb{C}}(n,k)$ of the fibre-wise self-adjoint operators on the tautological bundle T_k (resp. T_{n-k}).*

Let $\pi : P_k \oplus P_{n-k} \to Gr_{\mathbb{C}}(n,k)$ be the whitney sum of the two bundles. Each of its fiber is isomorphic to $S(k) \oplus S(n-k)$ when we pick a basis. There is a U(n) action on this bundle:

$$\begin{gathered} g \cdot (p, (p_k, p_{n-k})) = (g \cdot p, (g \circ p_k \circ g^{-1}, g \circ p_{n-k} \circ g^{-1})), \\ p \in Gr_{\mathbb{C}}(n,k), p_k \in P_k(p), p_{n-k} \in P_{n-k}(p). \end{gathered} \tag{4.13}$$

Suppose an open neighborhood $U \in S$ satisfies the spectral gap condition. Let $U(n) \cdot U$ be the group invariant neighborhood generated by U, that is,

$$U(n) \cdot U := \cup_{g \in U(n)} g \cdot U. \tag{4.14}$$

Then $U(n) \cdot U$ is open and connected, and also satisfies the spectral gap condition as U does, since U(n) action preserves the eigenvalues. From the proof of the Lemma 4.3, it is shown that in the neighborhood, the trivial $\mathbb{C}^n$ bundle over U naturally splits into two subbundles $E_k \oplus E_{n-k}$. And this gives a local product structure. We will prove that, for a U(n)-invariant neighborhood, there is actually a group equivariant isomorphism with the operator bundles defined above.

Lemma 4.4 (bundle map)**.** *If a point $X_0 \in S$ satisfies the spectral gap condition, then there is a neighborhood $X_0 \in V \subset S$ such that it is isomorphic to a neighborhood in the product of lower rank matrices and Grassmannian, i.e.*

$$\phi : V \cong V(k) \times V(n-k) \times V_{Gr} \subset S(k) \times S(n-k) \times Gr_{\mathbb{C}}(n,k) \subset P_n \oplus P_{n-k}.$$

Moreover, if we take the neighborhood $U(n) \cdot V$, it is isomorphic to a neighborhood $W \subset P_k \oplus P_{n-k}$ such that $\pi(W) = Gr_{\mathbb{C}}(n,k)$ and the isomorphism ϕ is $U(n)$-invariant.

Proof. From the proof of Lemma 4.3, there is a neighborhood U of X_0, such that each element $X \in U$ are decomposed into $L_X + R_X$. Moreover, it induces a decomposition of the trivial bundle $U \times \mathbb{C}^n$ into two subbundles:

$$U \times \mathbb{C}^n = E_k \oplus E_{n-k} \tag{4.15}$$

where $E_k(X)$ and $E_{n-k}(X)$ are determined by the projection operator P_X defined in equation (4.9):

$$E_k(X) = Im(P_X),\ E_{n-k}(X) = Im(P_X)^{\perp} \tag{4.16}$$

Let $(\xi_1, ...\xi_k)$ be the basis for $E_k(X_0)$. E_k over U is an open neighborhood in $Gr_{\mathbb{C}}(n,k)$. We can find a neighborhood V of X_0 (possibly smaller than U) such that, for every point in V, the k-dimensional space E_k projects onto $E_k(X_0)$. And an orthonormal basis of $E_k(X)$ is uniquely determined by requiring the projection of the first j vectors to $E_k(X_0)$ spans $(\xi_1, ...\xi_j)$ for every j smaller than k. In this way, we picked a basis for each fiber of E_k and E_k is trivialized to be a k-dimensional vector bundle on V. Since the action of X on $\mathbb{C}^n$ has been decomposed to L_X and R_X, then with the choice of basis, the action of L_X on $E_k(X)$ gives a $k \times k$ self-adjoint matrix, and by continuity, these matrices form a neighborhood V_k in S(k). And the same argument works for R_X.

Therefore, we have the following map ϕ:

$$\begin{aligned} \phi : V &\to P_k \oplus P_{n-k} \\ X &\mapsto (E_k(X), (L_X|_{E_k(X)}, R_X|_{E_{n-k}(X)})) \end{aligned} \tag{4.17}$$

This map is an isomorphism between V and $\phi(V)$. It's injective, since the action of the two invariant subspace uniquely determines the action on $\mathbb{C}^n$, therefore gives the unique operator X. The continuity of ϕ and ϕ^{-1} comes from the continuity of the projection operator defined in Lemma 4.1 therefore the continuity of E_k, L_X and R_X

are continuous.

Now take $U(n) \cdot V$, since E_k takes every possible k-subspace of $\mathbb{C}^n$ under the action of $U(n)$, we know that the first entry of $\phi(U(n) \cdot V)$ maps onto $Gr_{\mathbb{C}}(n,k)$. Moreover, since the decomposition respects the action of $U(n)$, it is easily seen that, for $g \in U(n), X \in G \cdot V$,

$$\phi(g \cdot X) = (g \cdot E_k(X), (g \circ L_X \circ g^{-1}, g \circ R_X \circ g^{-1})) = g \cdot (\phi(X)) \tag{4.18}$$

which means the isomorphism is group invariant. □

To do the induction, we will need to define an index on the inclusion isotropy types, so the blow up procedure could be done in the partial order given by the index. Recall that two matrices have the same isotropy type if they have the same "clustering" of eigenvalues. Now we define the isotropy index of a matrix X as follows.

Definition 4.5 (Isotropy index)**.** *Suppose the eigenvalues of a matrix X are*

$$\lambda_1 = .. = \lambda_{i_1} < \lambda_{i_1+1} = .. = \lambda_{i_2} < \lambda_{i_2+1} = ... < \lambda_{i_{k-1}+1} = ... = \lambda_n$$

then the isotropy index of X is defined as the set

$$I(X) = \{i_0 = 0, i_1, i_2, ..., i_{k-1}, i_k = n\}.$$

There is a partial order of this index on S, given by the inclusion of isotropy types. That is, if for matrix X and Y we have $I(X) \subset I(Y)$ then we say that the order is $X \leq Y$. Note there is an inverse inclusion to isotropy group. The smallest isotropy index is $I(\lambda I) = \{0, n\}$ while the isotropy group is $U(n)$ which is the largest. And the largest index is $\{0, 1, 2, ..., n-1, n\}$ which correspond to n distinct eigenvalues, where the isotropy group contains only identity.

The last lemma we need before the induction is the comparability of conjugacy class inclusion and the decomposition to two submatrices, which shows the order of resolution in Lemma 4.1 is comparable with the decomposition.

Lemma 4.5 (Compatibility with conjugacy class)**.** *The partial order of conjugacy class inclusion is comparable with the decomposition in Lemma 4.3.*

Proof. Suppose a neighborhood $V \subset S(n)$ has a decomposition in Lemma 4.3. We need to show that, if $S(n)^{I_1}$ is the stratum of minimal isotropy type in V, then the decomposition of this stratum corresponds to the minimal isotropy type in $U(k)$ and $U(n-k)$.

Since V satisfies the spectral gap condition, all the isotropy types in V would be subgroups of $U(k) \oplus U(n-k)$. Suppose the minimal stratum corresponds to the index $I = \{0, i_1, ..., i_m\}$ which must contain k as one element because of the spectral gap condition. Then the isotropy type of two subgroups are $\{0, i_1, ...k\}$ and $\{i_j - k = 0, j_{j+1}, ..., n-k\}$. They would still be the minimal in each subgroup, otherwise when the two smallest elements combined it'll give a smaller index than I which is a contradiction. □

Now we can finally prove Theorem 4.1 using the above lemmas.

Proof of Theorem 4.1. We prove the theorem by induction of the matrix size. The 2×2 case is shown in Lemma 4.2. Suppose the claim holds for all the cases up to $n-1$. Now we claim that, by an iterative blow up, we can get $\hat{S}(n)$ for dimension=n, with eigenvalues and eigenbundles lifted to satisfy the full eigenresolution properties.

As in the 2×2 example, we shall first consider the trace free slice $S_0(n)$ since other slices have the same behavior in terms of smoothness of eigenvalues and eigenbundles. Take the smallest index $I = \{0, n\}$ with the largest possible isotropy group $U(n)$, and the stratum in $S_0(n)$ with such an isotropy group is the zero matrix. After blowing up, we get $[S_0; 0]$ as the first step.

The next smallest index is $\{0, k, n\}$ where $1 < k < n$. And the strata corresponding to different k become disjoint in $[S_0; 0]$ because if the eigenvalues of a matrix $X \in S_0$ satisfy $k_1\lambda_1 + k_2\lambda_2 = 0, k_1'\lambda_1 + k_2'\lambda_2 = 0$, then $\lambda_1 = \lambda_2 = 0$, which has been blown up in the previous step. Therefore we can blow up those strata separately. For any point $X \in S_0(n)$ with $I(X) = \{0, k, n\}$, we can generate a neighborhood $U(n) \cdot V$

as in Lemma 4.4, which is isomorphic to a neighborhood in the bundle $P_k \oplus P_{n-k}$. Locally there is a product structure $V \cong V_k \times V_{n-k} \times V_{Gr} \subset S(k) \times S(n-k) \times Gr_{\mathbb{C}}(n,k)$. For every base point $p \in V_{Gr}$, since the fibre is isomorphic to a neighborhood in $S(k) \times S(n-k)$, the resolution can be done separately to V_k and V_{n-k}. And according to Lemma 4.5 the index order is preserved when decomposed into two subspaces, so the blow up construction indexed by isotropy type inclusion can be done on V_k and V_{n-k}. By induction, after the full resolution of the two subspaces, E_k and E_{n-k} both split into line bundles, and eigenvalues also extend to the frontface smoothly. And since this local product structure is U(n)-invariant on $U(n) \cdot V$, the splitting of eigenbundles are actually global.

Therefore, after the resolution, we have iteratively blown up the stratum according to isotropy indices to get

$$\hat{S} = [S; \{0\}; S^{I_1}; S^{I_2}; ...S^{I_n}], \tag{4.19}$$

above which there are n line bundles as eigenbundles and the corresponding eigenvalues are also smooth.

□

Bibliography

[1] Pierre Albin and Richard Melrose. Resolution of smooth group actions. volume 535 of *Contemp. Math.*, pages 1–26. Amer. Math. Soc., Providence, RI, 2011.

[2] E Bergshoeff, Ergin Sezgin, and Paul K Townsend. Supermembranes and eleven-dimensional supergravity. *Physics Letters B*, 189(1):75–78, 1987.

[3] Matthias Blau, Jose Figueroa-O'Farrill, and George Papadopoulos. Penrose limits, supergravity and brane dynamics. *Classical and Quantum Gravity*, 19(18):4753, 2002.

[4] Armand Borel and Lizhen Ji. *Compactifications of symmetric and locally symmetric spaces.* Springer, 2006.

[5] L Castellani, Riccardo D'Auria, P Fre, K Pilch, and P Van Nieuwenhuizen. The bosonic mass formula for Freund–Rubin solutions of d= 11 supergravity on general coset manifolds. *Classical and Quantum Gravity*, 1(4):339, 1984.

[6] E. Cremmer and J. Scherk. Spontaneous compactification of extra space dimensions. *Nuclear Phys. B*, 118(1–2):61–75, 1977.

[7] Erwann Delay. Essential spectrum of the Lichnerowicz laplacian on two tensors on asymptotically hyperbolic manifolds. *Journal of Geometry and Physics*, 43(1):33–44, 2002.

[8] S. K. Donaldson. Lefschetz fibrations in symplectic geometry. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, number Extra Vol. II, pages 309–314, 1998.

[9] Joel Fine. Constant scalar curvature Kähler metrics on fibred complex surfaces. *Journal of Differential Geometry*, 68(3):397–432, 2004.

[10] Joel Fine. *Constant scalar curvature Kähler metrics on fibred complex surfaces.* PhD thesis, University of London, 2004.

[11] Peter GO Freund and Mark A Rubin. Dynamics of dimensional reduction. *Physics Letters B*, 97(2):233–235, 1980.

[12] Terry Fuller. Hyperelliptic Lefschetz fibrations and branched covering spaces. *Pacific Journal of Mathematics*, 196(2):369–393, 2000.

[13] Robert E Gompf. A new construction of symplectic manifolds. *Annals of Mathematics*, pages 527–595, 1995.

[14] Robert E. Gompf and András Stipsicz. *4-manifolds and Kirby calculus*. Number 20. American Mathematical Soc., 1999.

[15] C. Robin Graham and John M. Lee. Einstein metrics with prescribed conformal infinity on the ball. *Adv. Math.*, 87(2):186–225, 1991.

[16] C. Robin Graham and Maciej Zworski. Scattering matrix in conformal geometry. *Invent. Math.*, 152(1):89–118, 2003.

[17] Daniel Grieser. Basics of the b-calculus. In *Approaches to singular analysis*, pages 30–84. Springer, 2001.

[18] Colin Guillarmou. Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds. *Duke Math. J.*, 129(1):1–37, 2005.

[19] Colin Guillarmou and Frédéric Naud. Wave 0-trace and length spectrum on convex co-compact hyperbolic manifolds. *Comm. Anal. Geom*, 14(5):945–967, 2006.

[20] Joshua M Kantor. *Eleven Dimensional Supergravity on Edge Manifolds*. PhD thesis, University of Washington, 2009.

[21] Dietrich Kramer and Ernst Schmutzer. *Exact solutions of Einstein's field equations*, volume 19. Cambridge University Press Cambridge, 1980.

[22] John M Lee. *Fredholm operators and Einstein metrics on conformally compact manifolds*. American Mathematical Soc., 2006.

[23] Rafe Mazzeo. The Hodge cohomology of a conformally compact metric. *Journal of differential geometry*, 28(2):309–339, 1988.

[24] Rafe Mazzeo. Elliptic theory of differential edge operators. I. *Comm. Partial Differential Equations*, 16(10):1615–1664, 1991.

[25] Rafe Mazzeo and Richard B Melrose. Analytic surgery and the eta invariant. *Geometric & Functional Analysis GAFA*, 5(1):14–75, 1995.

[26] Rafe Mazzeo and Richard B Melrose. Pseudodifferential operators on manifolds with fibred boundaries. *arXiv preprint math/9812120*, 1998.

[27] Rafe Mazzeo and Jan Swoboda. Asymptotics of the Weil–Petersson metric. *arXiv preprint arXiv:1503.02365*, 2015.

[28] Rafe Mazzeo and Boris Vertman. Elliptic theory of differential edge operators, II: boundary value problems. *ArXiv e-prints*, July 2013.

[29] Rafe R. Mazzeo and Richard B. Melrose. Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *J. Funct. Anal.*, 75(2):260–310, 1987.

[30] Richard Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4. AK Peters Wellesley, 1993.

[31] Richard Melrose and Xuwen Zhu. Resolution of the canonical fiber metrics for a lefschetz fibration. *arXiv preprint arXiv:1501.04124*, 2015.

[32] Richard B Melrose. Calculus of conormal distributions on manifolds with corners. *International Mathematics Research Notices*, 1992(3):51–61, 1992.

[33] Richard B. Melrose. *Geometric scattering theory*, volume 1. Cambridge University Press, 1995.

[34] Richard B Melrose. Differential analysis on manifolds with corners, 1996.

[35] Richard B Melrose, Paolo Piazza, et al. Families of Dirac operators, boundaries and the b-calculus. *J. Differential Geom*, 46(1):99–180, 1997.

[36] Boris Moishezon. *Complex surfaces and connected sums of complex projective planes*. Springer, 1977.

[37] Werner Nahm. Supersymmetries and their representations. *Nuclear Physics B*, 135(1):149–166, 1978.

[38] Horatiu Nastase. Introduction to supergravity. *arXiv preprint arXiv:1112.3502*, 2011.

[39] Kunio Obitsu and Scott A. Wolpert. Grafting hyperbolic metrics and Eisenstein series. *Math. Ann.*, 341(3):685–706, 2008.

[40] Joel W. Robbin and Dietmar A. Salamon. A construction of the Deligne-Mumford orbifold. *J. Eur. Math. Soc. (JEMS)*, 8(4):611–699, 2006.

[41] Robert Seeley and Isadore M Singer. Extending $\bar{\partial}$ to singular Riemann surfaces. *Journal of Geometry and Physics*, 5(1):121–136, 1988.

[42] Peter Van Nieuwenhuizen. Supergravity. *Physics Reports*, 68(4):189–398, 1981.

[43] Peter Van Nieuwenhuizen. The complete mass spectrum of d= 11 supergravity compactified on s4 and a general mass formula for arbitrary cosets m4. *Classical and Quantum Gravity*, 2(1):1, 1985.

[44] Fang Wang. Dirichlet-to-Neumann map for Poincare–Einstein metrics in even dimensions. *arXiv preprint arXiv:0905.2457*, 2009.

[45] Edward Witten. On flux quantization in M theory and the effective action. *J.Geom.Phys.*, 22:1–13, 1997.

[46] Scott A Wolpert. The hyperbolic metric and the geometry of the universal curve. *Journal of Differential Geometry*, 31(2):417–472, 1990.

[47] Xuwen Zhu. Eleven dimension supergravity equations on edge manifolds. *preprint*, 2014.

[48] Xuwen Zhu. Resolution of eigenvalues. *preprint*, 2014.