Eleven Dimension Supergravity Equations on Edge Manifolds

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Supergravity Equations

The 11 dimensional bosonic supergravity equations on $M = \mathbb{B}^7 \times \mathbb{S}^4$ are defined for a metric g and a 4-form F

$$\begin{cases} R_{\alpha\beta} & = \frac{1}{12}(F_{\alpha\gamma_1\gamma_2\gamma_3}F_\beta^{\gamma_1\gamma_2\gamma_3} - \frac{1}{12}F_{\gamma_1\gamma_2\gamma_3\gamma_4}F^{\gamma_1\gamma_2\gamma_3\gamma_4}g_{\alpha\beta}) \\ d * F & = -\frac{1}{2}F \wedge F \\ dF & = 0 \end{cases}$$

Result overview

All the smooth solutions in edge form with a given conformal infinity near the Freund-Rubin solution are parametrized by a 3-form and two functions on the bounding 6-sphere of $\mathbb{B}^7$.

Background: physics

- From the representation of super Lie algebra [Kac, 1978]
- Eleven is the maximal dimension in order to be physical [Nahm, 1975]
- Existence shown in [Cremmer-Scherk, 1977]
- Lower dimension cases can be obtained through dimension reductions that yield more smaller pieces [Nieuwenhuizen, 1985]
- Recent development in relation with AdS/CFT correspondence [Witten, 1997] [Blau-Figueroa-O'Farrill-Papadopoulos, 2002]
- Generalization of Einstein equations on a manifold

$$Ric(g) = (n-1)Rg$$

- Change of signature: Lorentzian to Riemannian by complexifying the time direction: $t \to i * t$

Derivation of Supergravity Equations

Euler-Lagrange equations from the following Lagrangian

$$L(g, A) = \int_M R dV_g - \frac{1}{2}\left(\int_M F \wedge *F + \int_M \frac{1}{3} A \wedge F \wedge F\right).$$

- Three terms: classical Einstein-Hilbert action, Yang-Mills type, Maxwell type
- A is a 3-form such that F is the field strength $F = dA$, need not to be globally defined

Supergravity Equations

The 11 dimensional bosonic supergravity equations on $M = \mathbb{B}^7 \times \mathbb{S}^4$ are defined for a metric g and a 4-form F

$$
\begin{aligned}
R_{\alpha\beta} &= \frac{1}{12}\left(F_{\alpha\gamma_1\gamma_2\gamma_3}F_{\beta}^{\gamma_1\gamma_2\gamma_3} - \frac{1}{12}F_{\gamma_1\gamma_2\gamma_3\gamma_4}F^{\gamma_1\gamma_2\gamma_3\gamma_4}g_{\alpha\beta}\right) \\
d * F &= -\frac{1}{2}F \wedge F \\
dF &= 0
\end{aligned}
$$

Remark

If $F = 0$, then any Einstein vacuum solutions would solve the equations.

A family of product solutions

X^7: Einstein manifold with negative scalar curvature $\alpha < 0$.
K^4: Einstein manifold with positive scalar curvature $\beta > 0$.
Consider $M = X \times K$, then

$$R_{\alpha\beta} = \begin{pmatrix} 6\alpha g^X_{AB} & 0 \\ 0 & 3\beta g^K_{ab} \end{pmatrix}$$

Let $F = c\,\mathrm{Vol}_K$. The contraction part is then

$$(F \circ F)_{\alpha\beta} = \frac{c^2}{12} \begin{pmatrix} -2g^X_{AB} & 0 \\ 0 & 4g^K_{ab} \end{pmatrix}$$

Therefore any set (c, α, β) satisfying

$$-c^2/6 = 6\alpha, c^2/3 = 3\beta$$

corresponds to a solution.

Freund-Rubin solution

The parameters

$$-c^2/6 = 6\alpha, c^2/3 = 3\beta$$

give a family of solutions $(X^7 \times K^4, c\,\mathrm{Vol}_K)$ with

$$\mathrm{Ric}(X^7) = 6\alpha g_X, \mathrm{Ric}(K^4) = 3\beta g_K.$$

In particular, when $c = 6$ there is the following product solution

Definition (Freund-Rubin solution)

The Freund-Rubin Solution is defined on $\mathbb{B}^7 \times \mathbb{S}^4$ as

$$(g_0, V_0) = (g_{\mathbb{H}^7} \times \frac{1}{4} g_{\mathbb{S}^4}, 6\,\mathrm{Vol}_{\mathbb{S}^4})$$

Review: Poincaré-Einstein metric on the n-ball

Poincaré-Einstein metric: Einstein metric with a conformal infinity.

Definition (Conformal infinity)

For $\hat{g}$ a metric on ∂M, we say a metric g on M has a conformal infinity $[\hat{g}]$ if for a boundary defining function x, $x^2 g$ extends to $\bar{M}$ and is conformal to $\hat{g}$ on ∂M.

Existence of Poincaré-Einstein metric near the standard hyperbolic metric on $\mathbb{B}^{n+1}$:

Theorem [Graham-Lee, 1991]

Let $\hat{h}$ be the standard metric on $\mathbb{S}^n$. For any smooth Riemannian metric $\hat{g}$ on $\mathbb{S}^n$ which is sufficiently close to $\hat{h}$ in $C^{2,\alpha}$ norm if $n > 4$ or $C^{3,\alpha}$ norm if $n = 3$, for some $0 < \alpha < 1$, there exists a smooth metric g on the interior of M, with conformal infinity $[\hat{g}]$ and satisfies $\mathrm{Ric}(g) = -ng$.

Review of the Poincaré-Einstein Proof

Proof.

- Use DeTurck term $\phi(t, g)$ to break the gauge
- Compute the linearization of the operator $Q = \mathrm{Ric} - \phi(t, g) + n$
- Using indicial roots computation and Cheng-Yau Maximum Principle [Cheng-Yau, 1980] to show that dQ is an isomorphism between weighted Sobolev spaces
- Then use a perturbation argument to show that the nonlinear operator Q is a bijection between the boundary parameter and the Einstein metric.

□

Edge manifold

Edge operator theory [Mazzeo, 1991] is introduced on a compact manifold M with boundary, where the boundary has a fibration structure:

$$\begin{array}{ccc} F & \longrightarrow & \partial M \\ & & \downarrow \pi \\ & & B \end{array}$$

- Edge vector fields are regular in the interior and parallel to fibers on the boundary. In local coordinates, Lie algebra $\mathcal{V}_e$ is spanned by $\{x\partial_x, x\partial_{y^I}, \partial_{z^j}\}$.
- By duality, edge bundles are given by tensor products of $\{dx/x$, $dy^I/x, dz^j\}$
- The edge metric is of the form

$$g = a_{00}\frac{dx^2}{x^2} + a_{0I}\frac{dxdy^I}{x^2} + a_{IJ}\frac{dy^I dy^J}{x^2} + a_{0j}\frac{dxdz^j}{x} + a_{Ij}\frac{dy^I dz^j}{x} + a_{ij}dz^i dz^j.$$

Edge Sobolev Space

Definition (Edge Sobolev space)

On an edge manifold M, the edge Sobolev space $H_e^s(M)$ is defined as

$$H_e^s(M) = \{u \in L^2(M) | V_e^k u \in L^2(M), V_e \in \mathcal{V}_e(M), 0 \leq k \leq s\}$$

Another related Lie algebra is the b-vector field $\mathcal{V}_b$ [Melrose, 1992], which is tangent to the boundary, and spanned locally by $\{x\partial_x, \partial_y, \partial_z\}$.

Definition (Weighted hybrid Sobolev space)

$$x^\delta H_{e,b}^{s,k}(M) = \{x^\delta u \in H_e^s(M) | V_b^i u \in H_e^s(M), V_b \in \mathcal{V}_b, 0 \leq i \leq k\}$$

Semi-Fredholm property of elliptic edge operator

- Elliptic edge operator $L = \sum_{j+|\alpha|+|\beta| \leq m} a_{j,\alpha,\beta}(x\partial_x)^j (x\partial_y)^\alpha (\partial_z)^\beta$ is defined for the principal edge symbol

$$\sum_{j+|\alpha|+|\beta|=m} a_{j,\alpha,\beta}\xi_1^j \xi_2^\alpha \xi_3^\beta$$

- Example: Laplacian on $\mathbb{H}^2 \times \mathbb{S}^1$

Theorem [Mazzeo, 1991]

If an elliptic edge operator $L \in \mathrm{Diff}_e^m(M)$ has two properties:
(a) Constant indicial roots over the boundary;
(b) Its normal operator L_0 and adjoint L_0^t has the unique continuation property,
then L is essentially injective (resp. surjective) on $x^\delta H_e^k(M)$ for a weight parameter $\delta \gg 0$ (resp. $\delta \ll 0$) with $\delta \notin \Lambda = \{\mathrm{Re}\,\theta + 1/2 : \theta \in spec_b L\}$, and in either case has closed range.

Main theorem

Define three bundles over $\mathbb{S}^6 = \partial\mathbb{B}^7$:

$$V_1 := \{v_1 \in C^\infty(\mathbb{S}^6; \bigwedge^3 T^*\mathbb{S}^6) : *_{\mathbb{S}^6} v_1 = iv_1\}.$$

$$V_2 := \{v_2 \otimes \xi_{16} : v_2 \in C^\infty(\mathbb{S}^6; \mathbb{R}), \xi_{16} \in E^{cl}_{16}(\mathbb{S}^4)\}$$

$$V_3 := \{v_3 \otimes \xi_{40} : v_3 \in C^\infty(\mathbb{S}^6; \mathbb{R}), \xi_{40} \in E^{cl}_{40}(\mathbb{S}^4)\}$$

Theorem

For $k \gg 0, \delta \in (0,1), s \geq 2$, fixing a conformal infinity $[\hat{h}]$ that is close to $[\hat{g}_0]$ at the boundary $\mathbb{S}^6$, then given any smooth section $v = (v_1, v_2, v_3)$ of the bundle $\oplus_{i=1}^3 V_i$ with a sufficiently small H^k norm, there is a unique $(g, F) \in \mathcal{C}^\infty(M; \mathrm{Sym}^2({}^eT^*M) \oplus {}^e\bigwedge^4 T^*M)$ such that
(a) $(g - g_0, F - V_0) \in x^{-\delta} H^{s,k}_{e,b}(M; W)$ and has a leading expansion given by v;
(b) (g, F) satisfy the supergravity equation $S(g, F) = 0$ with g having the conformal infinity $[\hat{h}]$.

Gauge term

We apply the DeTurck type term: $\phi_{(g,t)} = \delta_t^*(tg)^{-1}\delta_t G_t g$ to get the full nonlinear system

$$Q : S^2(T^*M) \oplus \bigwedge_{cl}^4(M) \to S^2(T^*M) \oplus \bigwedge_{cl}^4(M)$$

$$\begin{pmatrix} g \\ F \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{Ric}(g) - \phi_{(g,t)} - F \circ F \\ d * (d * F + \frac{1}{2} F \wedge F) \end{pmatrix}$$

Then we get the following gauge elimination:

Proposition (Gauge elimination lemma)

If (k,H) satisfies the linearized equation $dQ_g(k, H) = 0$, then we can find a 1-form v and $\tilde{k} = k + L_{v^\sharp} g$ such that $dS_g(\tilde{k}, H) = 0$.

Linearization of the gauged operator

The operator $Q : W \to W$ has the following linearization at the point (g_0, V_0):

$$dQ_{g_0,V_0} : \Gamma(\mathrm{Sym}^2({}^eT^*M) \oplus {}^e\bigwedge^4(M)) \to \Gamma(\mathrm{Sym}^2({}^eT^*M) \oplus {}^e\bigwedge^4(M))$$

$$\begin{pmatrix} k \\ H \end{pmatrix} \mapsto \begin{pmatrix} \Delta k + \mathrm{LOT} \\ d * (d * H + 6V_S \wedge H + 6d *_H k_{1,1} + 3d(7\sigma - 4\tau) \wedge V_H) \end{pmatrix}$$

where the lower order term matrix LOT is as follows:

$$\mathrm{LOT} = \begin{pmatrix} \begin{matrix} -k_{IJ} - 6Tr_S(k)t_{IJ} \\ +Tr_H(k)t_{IJ} + 2 *_S H_{(0,4)}t_{IJ} \end{matrix} & 6k_{1,1} - 3 *_S H_{1,3} \\ 6k_{1,1} - 3 *_S H_{1,3} & \begin{matrix} 4k_{ij} + 8Tr_S(k)t_{ij} \\ - *_S H_{0,4}t_{ij} \end{matrix} \end{pmatrix}$$

Splitting with the product structure

- From the form equation

$$6d_H *_7 k_{(1,1)} + 3d_S(Tr_{H^7}(k) - Tr_{S^4}(k)) \wedge^7 V + d_S * H_{(0,4)} + d_H * H_{(1,3)} = 0$$

$$d_S * H_{(1,3)} + d_H * H_{(2,2)} + 6d_S *_7 k_{(1,1)} = 0$$

$$d_S * H_{(2,2)} + d_H * H_{(3,1)} = 0$$

$$d_S * H_{(3,1)} + d_H * H_{(4,0)} + W \wedge H_{(4,0)} = 0$$

- From the metric equation:

$$\frac{1}{2}\triangle_s k_{Ij} + \frac{1}{2}\triangle_H k_{Ij} + 6k_{Ij} - 3 *_S H_{(1,3)} = 0$$

$$\frac{1}{2}(\triangle_s + \triangle_H)k_{IJ} - k_{IJ} - 6\,Tr_S(k)t_{IJ} + Tr_H(k)t_{IJ} + 2H_{(0,4)}t_{IJ} = 0$$

$$\frac{1}{2}(\triangle_S + \triangle_H)k_{ij} + 4k_{ij} + 8\,Tr_S(k)t_{ij} - H_{(0,4)}t_{ij} = 0$$

Inidicial operator: introduction

Definition (Indicial operator)

Let $L : \Gamma(E_1) \to \Gamma(E_2)$ be an edge operator between two vector bundles over M. For any boundary point $p \in B$, and $s \in \mathbb{C}$, the indicial operator of L at point p is defined as

$$I_p[L](s) : \Gamma(E_1|_{\pi^{-1}(p)}) \to \Gamma(E_2|_{\pi^{-1}(p)})$$

$$(I_p[L](s))v = x^{-s} L(x^s \tilde{v})|_{\pi^{-1}(p)}$$

where $\tilde{v}$ is an extension of v to a neighborhood of $\pi^{-1}(p)$. The indicial roots of L at point p are those $s \in \mathbb{C}$ such that $I_p[L](s)$ has a nontrivial kernel. And the corresponding kernels are called indicial kernels.

Inidicial operator computation for dQ

- In general, indicial operators are partial differential operators. However for conformally compact case $I_p[L](s)$ is a bundle map
- Intuition: leading expansions of solutions for ODE
- Example: Laplacian on hyperbolic space $\mathbb{H}^{n+1}$, consider the solution to

$$[\Delta - \alpha(n-\alpha)]f = 0$$

with asymptotic $x^{-\beta} f|_{\partial M} = h$. The indicial operator in this case is

$$I[L](s) = -s^2 + ns - \alpha(n-\alpha)$$

which gives indicial roots $s = \alpha$ and $s = n - \alpha$. h and β must satisfiy

$$[-\beta^2 + n\beta - \alpha(n-\alpha)]h = 0$$

which means either $h = 0$ or β must be one of the indicial roots.
- So the indicial operator determines what order the asymptotic expansion the solution can take.

Hodge Decomposition on $\mathbb{S}^4$

- The linear operator commutes with $\Delta_{\mathbb{S}^4}$
- dQ projects to eigenspaces

$$dQ = \sum_{\lambda \geq 0} dQ^{\lambda} := \sum_{\lambda} \pi_{\lambda} \circ dQ \circ \pi_{\lambda}$$

- Eigenvalues on 4-sphere (rescaled by 4 due to the metric)
  - Eigenvalues on functions: $4k(k+3)$;
  - Eigenvalues on closed one-forms: $4(k+1)(k+4)$;
  - Eigenvalues on coclosed one-forms: $4(k+2)(k+3)$.

Indicial roots result

With the Hodge decomposition, the linear blocks decompose further. And the indicial roots are computed for each subspace.

Proposition

(a) The indicial roots of the linearized equations appear in pairs, symmetric around the line $\Re z = 3$*.*
(b) They are separated away from L^2*, except three pairs*

- $\theta_1 = 3 \pm 6i$, *indicial kernel* V_1
- $\theta_2 = 3 \pm i\sqrt{21116145}/1655$, *indicial kernel* V_2
- $\theta_3 = 3 \pm i3\sqrt{582842}/20098$, *indicial kernel* V_3

Indicial roots not on the L^2 line

Proposition

For any weight $s \in \mathbb{R}$ and any orders k,l, the bounded operator defined as

$$dQ : x^s H_{e,b}^{k+2,l}(\mathbb{B}^7 \times \mathbb{S}^4; W) \to x^s H_{e,b}^{k,l}(\mathbb{B}^7 \times \mathbb{S}^4; W)$$

is such that $\pi_{\geq\lambda} dQ$ is an isomorphism onto the range of $\pi_{\geq\lambda}$ for some $\lambda \in [0, \infty)$ (depending on s but not on k and l).

- Using [Mazzeo, 1991]: parametrix construction for elliptic edge operator
$$\mathrm{Id} - dQ \circ E,\ \mathrm{Id} - E \circ dQ \in \Psi_e^{-\infty}(M; W)$$
- Residue bounded: for some $s, p, k, \Psi_e^{-\infty} \subset x^s H_{e,b}^{p,k}(M)$ which is contained in an L^2 space.
- Then use Plancherel to show decay of $R^{\geq\lambda}$ for large λ, which shows $(dQ \circ E)^{\geq\lambda}$ an isomorphism.

Individual eigenvalues not on the L^2 line

Lemma

For $\lambda > 40$, $dQ^\lambda : \pi_\lambda x^\delta H^{s,k}_{e,b}(M; W) \to \pi_\lambda x^\delta H^{s-2,k}_{e,b}(M; W)$ is Fredholm for $\delta \in (-1, 1)$ and any s, k. And for $\delta > 0$, dQ^λ is injective on $\pi_\lambda x^\delta H^{s,k}_{e,b}(M; W)$ and surjective on $\pi_\lambda x^{-\delta} H^{s,k}_{e,b}(M; W)$.

- Use normal operator (ODE) as an isomorphism
- No finite dimensional L^2 eigenspaces for functions and tensors on $\mathbb{H}^7$.

Scattering on $\mathbb{H}^n$

The scattering matrix of (X, g) is a meromorphic family S(s) of pseudodifferential operators on X defined in terms of the behaviour at infinity of solutions of $(\Delta_g - s(n-s)]u = 0$. More explicitly, consider the solution

$$[\Delta_g - s(n-s)]u = 0, u = Fx^{n-s} + Gx^s$$

then the scattering matrix $S(s)$ is defined as the operator

$$S(s) : F|_{\partial X} \to G|_{\partial X}.$$

Graham and Zworski [Graham-Zworski, 2003] gave the description of scattering matrix in hyperbolic space:

$$S(s) = 2^{n-2s}\frac{\Gamma(n/2-s)}{\Gamma(s-n/2)}\Delta_{S^6}^{s-n/2},$$

Individual eigenvalues on the L^2 line

For $\lambda = 0, 16, 40$, we construct two generalized inverses

$$R_{\pm}^{\lambda} = \lim_{\epsilon \to 0} (dQ^{\lambda} \pm i\epsilon)^{-1}.$$

- $R_+ \circ dQ = R_- \circ dQ = Id$.
- $(R_+ - R_-)$ characterize the real-valued kernel in the base case.
- Transversality: use $(R_+ + R_-)$ to give a real-valued parametrization in implicit function theorem.

Implicit function theorem: Domain

Definition (Domain of nonlinear operator)

For a Poincaré-Einstein metric h that is close to the base hyperbolic metric and a set of parameters $v = (v_1, v_2, v_3)$ in bundle V, the domain $D_{h,v}$ of the nonlinear operator is defined as

$$D_{h,v} := \{\frac{1}{2}(R_+ + R_-)f + Pv : f \in x^\delta H^{0,k}_{e,b}(M; W)\}.$$

Then we show that the nonlinear terms are all of lower order:

Lemma

The product type nonlinear terms: $F \circ F - d(F \circ F)$, $F \wedge F - d(F \wedge F)$, *and* $\mathrm{Ric} - d(\mathrm{Ric})$ *are all contained in* $x^\delta H^{2,k}_{e,b}(M; W)$.

Using the fact that, for k large enough and $r \geq -3$, $x^r H^{s,k}_{e,b}(M)$ is an algebra.

Implicit function theorem

Consider the map

$$Q_{h,v} \cdot \circ (dQ_{0,0})^{-1} : \oplus_{i=1}^{3} V_i \times x^\delta H_{e,b}^{0,k}(M; W) \to x^\delta H_{e,b}^{0,k}(M; W)$$

$$(v, f) \mapsto Q_{h,v} \circ (dQ_{0,0})^{-1}(f)$$

Using implicit function theorem, there is a continuous differentiable map $g : U_1 \to U_2$ for $U_1 \subset \oplus V_i$, $U_2 \subset x^\delta H_{e,b}^{0,k}(M; W)$ such that

$$Q_{h,v} \cdot (dQ_{0,0})^{-1}(g(v)) = 0.$$

Theorem

For any $\delta \in (0,1), s \geq 2, k \gg 0$ *there exists* $\rho > 0$*, such that, for a Poincaré-Einstein metric h that is sufficiently close to the base metric* g_0*, for each small boundary value perturbation* $v = \oplus_{i=1}^{3} v_i^+$ *with* $\|v\|_{H_b^k(M;\oplus V_i)} < \rho$*, there is a unique solution* $(g,F) \in D_{v,h} \subset x^{-\delta} H_{e,b}^{s,k}(M;W)$ *satisfying the supergravity equations* $S(g,F) = 0$ *with the following leading expansion*

$$(g,F) = (h, 6\,\mathrm{Vol}_{\mathbb{S}^4}) + \sum_{i=1}^{3} v_i^+ \xi_i x^{\theta_i^+} + v_i^+ S_i(v_i^+) \xi_i x^{\theta_i^-}$$

Some further questions

1. Physics meaning of the three pairs of parameters in physics (Change of gravity, change of field strength)
2. The metric is Riemannian, what about the Lorentzian metric case, i.e. $AdS^7 \times \mathbb{S}^4$?
3. Instead of representing the spacetime by $\mathbb{S}^4$, one other family of solutions are given on $AdS^4 \times \mathbb{S}^7$, any other solutions near that point?

Thank you for your attention!