Constant curvature conical metrics

Xuwen Zhu

Joint with Rafe Mazzeo and Bin Xu

Outline

1. Uniformization with conical singularities
2. Deformation rigidity
3. Compactified configuration space

Constant curvature metrics on Riemann surfaces

- Classical uniformization theorem: for a given Riemann surface, there is a unique (smooth) constant curvature metric

$$\text{(Gauss–Bonnet)} \qquad \chi(M) = \frac{1}{2\pi} KA$$

$$\chi(M) = \text{ Euler characteristic }, \; K = \text{ curvature }, \; A = \text{ area}$$

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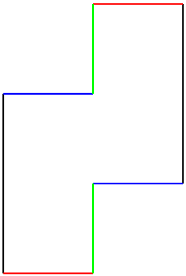
- Near a cone point with angle $2\pi\beta$, in geodesic polar coordinates

$$g = \begin{cases} dr^2 + \beta^2 r^2 d\theta^2 & K = 0 \quad \text{(flat)} \\ dr^2 + \beta^2 \sin^2 r d\theta^2 & K = 1 \quad \text{(spherical)} \\ dr^2 + \beta^2 \sinh^2 r d\theta^2 & K = -1 \quad \text{(hyperbolic)} \end{cases}$$

- In conformal coordinates $z = (\beta r)^{1/\beta} e^{i\theta}$, $\quad g = f(z)|z|^{2(\beta-1)}|dz|^2$

Some examples of constant curvature conical metrics



Translation surfaces
$(K = 0)$

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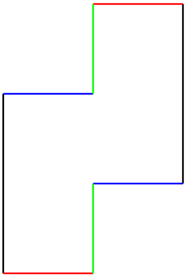

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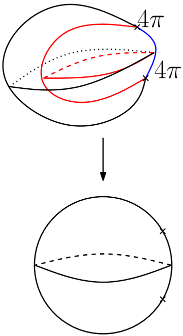

Branched covers of constant curvature surfaces ($K = -1, 0, 1$)

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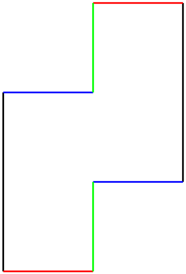

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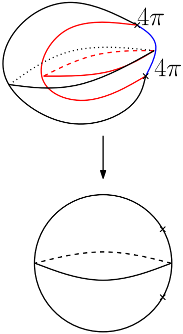

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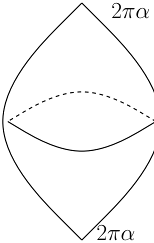

Spherical footballs ($K = 1$)

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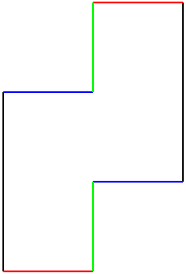

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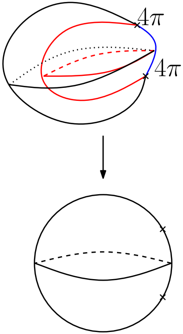

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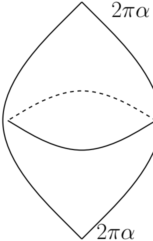

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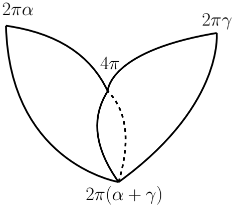

"Heart": footballs glued along geodesics $(K = 1)$

The study of constant curvature conical metrics is related to:

- Magnetic vortices: solitons of gauged sigma-models on a Riemann surface
- Mean Field Equations: models of electro-magnetism
- Toda system: multi-dimensional version
- Higher dimensional analogue: Kähler–Einstein metrics with conical singularities
- Bridge between the (pointed) Riemann moduli spaces: cone angle from 0 to 2π

This subject can be approached in many ways:

- PDE: singular Liouville equations
- Complex analysis: developing maps and Schwarzian derivatives
- Synthetic geometry: cut-and-glue

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A singular uniformization problem

Consider the following "conical data":

- n distinct points $\mathfrak{p} = (p_1, \ldots, p_n)$
- Angle data $\vec{\beta} = (\beta_1, \ldots, \beta_n),\ \beta_i \in \mathbb{R}^+ \setminus \{1\}$
- Conformal structure $\mathfrak{c}$ given by the underlying Riemann surface

Question

Given conical data $(\mathfrak{p}, \vec{\beta}, \mathfrak{c})$, does there exist a unique constant curvature conical metric with this data?

When uniformization holds

Theorem (Heins '62, McOwen '88, Troyanov '91, Luo–Tian '92)

For any compact Riemann surface $(M, \mathfrak{c})$ *and conical data* $(\mathfrak{p}, \vec{\beta})$ *with*

- $\chi(M, \vec{\beta}) \leq 0$; *or*
- $\chi(M, \vec{\beta}) > 0$, $\vec{\beta} \in T \subset (0,1)^k$ *where* T *is the Troyanov region*

there is a unique constant curvature conical metric with this data.

Theorem (Mazzeo–Weiss '15)

If $\vec{\beta} \in (0,1)^k$, *then there is a well-defined* $(6\gamma - 6 + 3k)$*-dimensional moduli space.*

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Spherical metrics with large cone angles

- The remaining case: $\chi(M, \vec{\beta}) > 0$, at least one of the angles greater than 2π
- Uniformization fails in this case
- Existence: constraints on conical data $(\mathfrak{p}, \vec{\beta}, \mathfrak{c})$
  Mondello–Panov '16, Chen–Lin '17, Chen–Kuo–Lin–Wang '18…
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  Chen–Wang–Wu–Xu '14, Eremenko '17,
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- Deformation: obstructions exist [Z '19]
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Outline of the main result

Our results provide new understanding of the local structure of the moduli space where it is not smoothly parametrized:

Theorem (Mazzeo–Z '19)

- *The local deformation with respect to* $(\mathfrak{c}, \mathfrak{p}, \vec{\beta})$ *has rigidity precisely when* $2 \in \mathrm{Spec}(\Delta_g^{\mathrm{Fr}})$*;*
- *It can be "desingularized" by adding more coordinates via splitting of cone points.*

- Understanding this problem through a nonlinear PDE:

$$\{\text{Constant curvature } K \text{ conical metrics}\}$$
$$\updownarrow$$
$$\left\{ \begin{array}{c} \text{Solutions to the Liouville equation} \\ \Delta_{g_0} u - K e^{2u} + K_{g_0} = 0 \end{array} \right\}$$

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Setup

- From now on we study spherical metrics ($K = 1$)
- We fix the Riemann surface $(M, \mathfrak{c})$ and do not vary cone angles
- $\mathscr{U}(\vec{\beta})$: the space of all cone metrics (not necessarily spherical) with cone angles $\vec{\beta} \in \mathbb{R}^n$
- $\mathbf{p} : \mathscr{U}(\vec{\beta}) \to M^n$ the positions of the cone points
- $\mathcal{S}(\vec{\beta}) \subset \mathscr{U}(\vec{\beta})$: the set of spherical cone metrics
- In general $\mathbf{p} : \mathcal{S}(\vec{\beta}) \to M^n$ is not a local diffeomorphism: we cannot parametrize elements of $\mathcal{S}(\vec{\beta})$ by cone point positions [Z '19]
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Deformation and linear obstructions

- Fix $g_0 \in \mathcal{S}(\vec{\beta})$. We study local deformations $g_t : (-\epsilon, \epsilon) \to \mathcal{S}(\vec{\beta})$ and cone point positions $\mathfrak{p}_t = \mathbf{p}(g_t)$.
- We have $g_t = e^{2u_t} g_0$, where u_t satisfies $u_0 = 0$ and solves the singular Liouville equation

$$\Delta_{g_0} u_t - e^{2u_t} + 1 = 0,$$

  Linearized equation: $(\Delta_{g_0} - 2)v = 0$ where $v := \partial_t u_t|_{t=0}$
- If $v \in \ker(\Delta^{\mathrm{Fr}}_{g_0} - 2)$ where $\Delta^{\mathrm{Fr}}_{g_0}$ is the Friedrichs Laplacian, then $\partial_t \mathfrak{p}_t|_{t=0} = 0$: obstruction to injectivity of $\mathbf{p}$.
- $\partial_t \mathfrak{p}_t|_{t=0}$ gives the singular terms of v (those not in the Friedrichs domain). If $\ker(\Delta^{\mathrm{Fr}}_{g_0} - 2) \neq 0$ then it might be impossible to find a solution with given singular terms: obstruction to surjectivity of $\mathbf{p}$.
- We say $\vec{A} (= \partial_t \mathfrak{p}_t|_{t=0})$ satisfies linear constraints if there exists a solution v to $(\Delta_{g_0} - 2)v = 0$ with singular terms prescribed by $\vec{A}$.

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Is 2 an eigenvalue of Δ_g^{Fr}?

- When $\vec{\beta} \in (0,1)^k$: the only spherical metrics with eigenvalue 2 are footballs (Bochner's technique / integration by parts)
- When at least one $\beta_i > 1$: the argument would not work any more
- Examples of metrics with $2 \in \mathrm{Spec}(\Delta_g^{\mathrm{Fr}})$: footballs, "heart", branched covers of the standard sphere
- Metrics with reducible monodromy all satisfy $2 \in \mathrm{Spec}(\Delta_g^{\mathrm{Fr}})$
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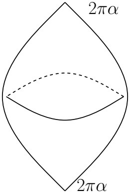

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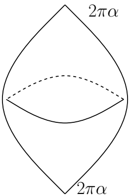

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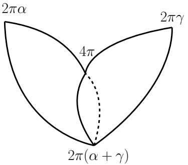
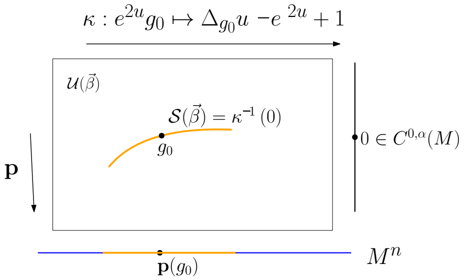


- The eigenfunctions on two footballs glue to a good eigenfunction ψ
- The complex gradient vector field of ψ again corresponds to conformal dilations
- This generates a family of spherical metrics with the same $\vec{\beta}$
- Rigidity: this family gives all spherical metrics with such $\vec{\beta}$ [Z '19]

A schematic picture



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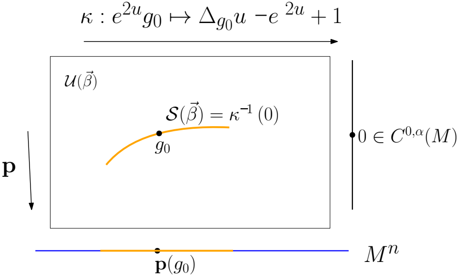

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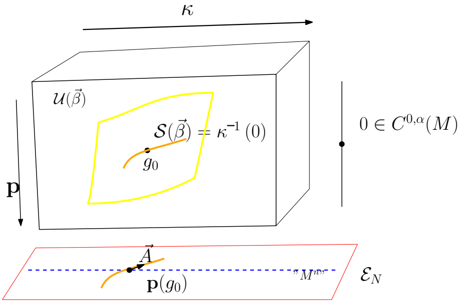

When 2 $\in \mathrm{Spec}(\Delta^{\mathrm{Fr}}_{g_0})$, in order to get a surjective map, we need to enlarge the parameter space to include splitting

A trichotomy theorem

Theorem (Mazzeo–Z, '19)

Let (M, g_0) be a spherical conic metric. Let $N = \sum_{j=1}^{k} \max\{[\beta_j], 1\}$. Let ℓ be the multiplicity of the eigenspace of $\Delta_{g_0}^{\mathrm{Fr}}$ with eigenvalue 2. There are three cases: $\ell = 0, 1 \leq \ell < 2N, \ell = 2N$.

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3. *(Complete rigidity) If* $\ell = 2N$*, then there is no nearby spherical cone metric obtained by moving or splitting the cone points of* g_0*.*

Cone points collision

- To set up the nonlinear analysis, one needs to understand the splitting (or merging) of cone points
- We developed an $\mathcal{C}^\infty$ model that encodes information of such behaviors for all constant curvature conical metrics (not only spherical)
- Scale back the distance between two cone points (“blow up”)

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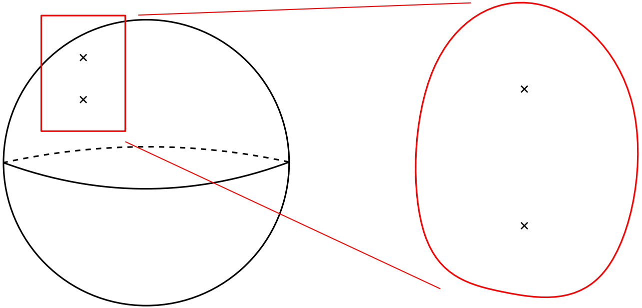
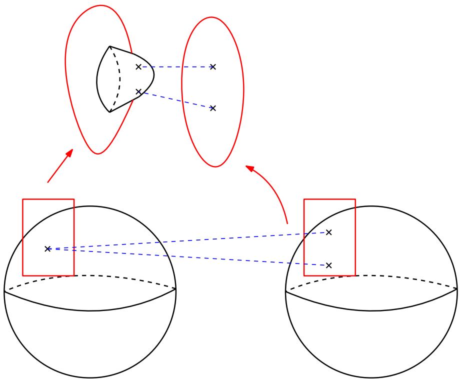


When two points collide

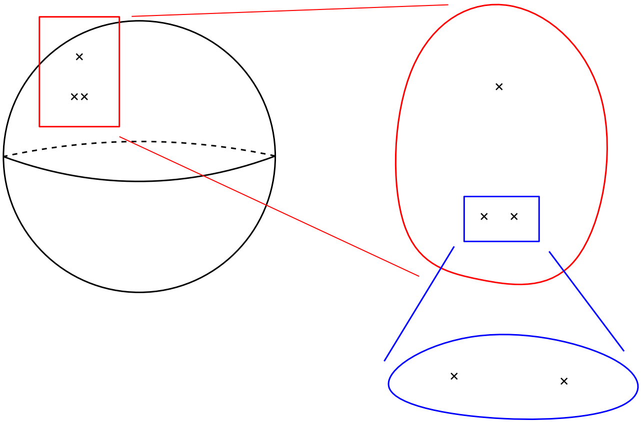
- Scale back the distance between two cone points (“blow up”)
- Half sphere at the collision point, with two cone points over the half sphere:



- Flat metric on the half sphere, and curvature K metric on the original surface

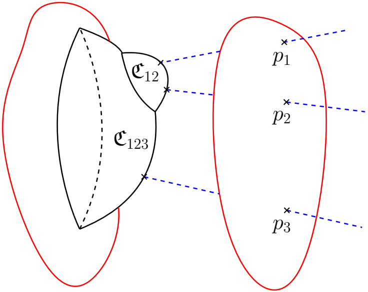
Iterative structure

- When there are several levels of distance: scale iteratively



Iterative structure

- "bubble over bubble" structure
- Higher codimensional faces from deeper scaling
- Flat conical metrics on all the new faces



- Iterative singular structures:
  Albin & Leichtnam & Mazzeo & Piazza '09-'19,
  Degeratu–Mazzeo '14, Kottke–Singer '15-'18,
  Albin–Gell-Redman '17, Albin–Dimakis–Melrose '19, ...

Resolution of the configuration space

This "bubbling" process can be expressed in terms of blow-up of product $M^k \times M \to M^k$ ($k = 2$ in the picture)

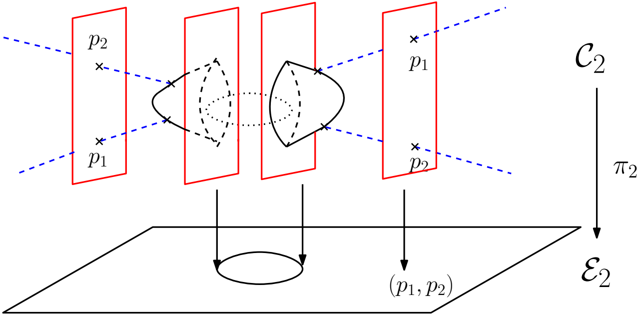


Figure: "Centered" projection of $\mathcal{C}_2 \to \mathcal{E}_2$

Results about fiber metrics

Theorem (Mazzeo–Z '17)

For any given $\vec{\beta}$, the family of constant curvature metrics with conical singularities is polyhomogeneous on this resolved space.*

- *The metric family can be hyperbolic / flat (with any cone angles), or spherical (with angles less than 2π, except footballs)
- Solving the curvature equation uniformly

$$\Delta_{g_0} u - Ke^{2u} + K_{g_0} = 0$$

- The bubbles with flat conical metrics represent the asymptotic properties of merging cones
- We then applied this machinery to understand the big cone angle case [Mazzeo–Z '19]

Linear constraints given by eigenfunctions

- The splitting creates extra dimensions, which fills up the cokernel of the linearized operator $\Delta_g^{\mathrm{Fr}} - 2$
- The direction of admissible splitting is determined by the expansion of the eigenfunctions
- Recall $N = \sum_{j=1}^{k} \max\{[\beta_j], 1\}$. An eigenfunction gives a $2N$-tuple $\vec{b}$
- The tangent of splitting directions are given by vectors $\vec{A}$ that are orthogonal to all such $\vec{b}$ (linear constraints)
- The bigger dimension of eigenspace, the more constraint on the direction of splitting
- How to get the splitting direction from $\vec{A}$: “almost” factorizing polynomial equations

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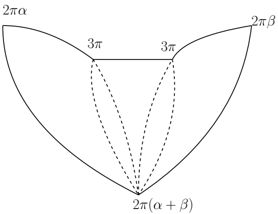
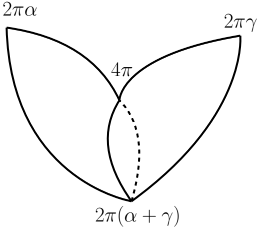
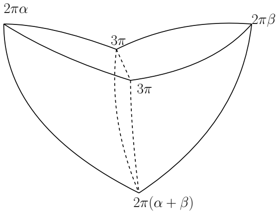
An example: open-heart surgery

- We obtain a deformation rigidity for the "heart"
- The cone point with angle 4π is split into two separate points
- In the equal splitting case: $4\pi \to (3\pi, 3\pi)$
- The spectral data dictates which splitting is possible:



Yes





No

Thank you for your attention!