Spectral properties of reducible conical metrics

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Outline

Constant curvature metrics with conical singularities

- A constant curvature metric with prescribed conical singularities is a smooth metric with constant curvature, except near p_j the metric is asymptotic to a cone with angle $2\pi\beta_j$.

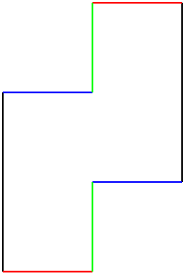
$$\text{(Gauss–Bonnet)} \quad \chi(\Sigma, \vec{\beta}) := \chi(\Sigma) + \sum_{j=1}^{k} (\beta_j - 1) = \frac{1}{2\pi} KA$$

- Locally near a cone point with angle $2\pi\beta$, written in geodesic polar coordinates

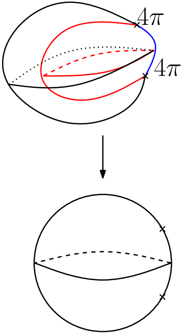
$$g = \begin{cases} dr^2 + \beta^2 r^2 d\theta^2 & K = 0; \\ dr^2 + \beta^2 \sin^2 r d\theta^2 & K = 1; \\ dr^2 + \beta^2 \sinh^2 r d\theta^2 & K = -1 \end{cases}$$

- Conformal coordinates: $f(z)|z|^{2(\beta-1)}|dz|^2$

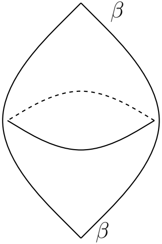
A few special examples



Translation surfaces

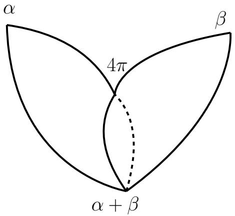


Branched covers of constant curvature surfaces

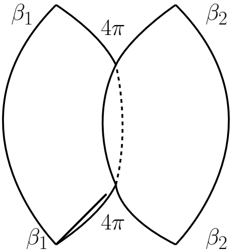


Spherical footballs

A few more examples



Two footballs glued along geodesics



Another kind of gluing

The uniformization problem: the PDE approach

Question

Given conical data $(\Sigma, \mathfrak{p}, \vec{\beta})$ satisfying the Gauss–Bonnet condition, does there exist a constant curvature conical metric?

- In the hyperbolic or flat case, yes.
  [Heins, '62] [Troyanov, '86] [McOwen, '88]
- In the spherical case, not always.
  [Troyanov, '91] [Mondello–Panov, '16-'19]
- We would like to understand it through a PDE:

$$\{\text{Spherical conical metrics}\}$$
$$\updownarrow$$
$$\left\{ \begin{array}{c} \text{Solutions to the Liouville equation} \\ \Delta_{g_0} u - K e^{2u} + K_{g_0} = 0 \end{array} \right\}$$

Here g_0 is either a smooth metric (then u has singularities); or a conical metric with the given conical data (then u is bounded).

The linearized operator

- The linearized operator is given by $\Delta_g - 2$ at a spherical conical metric g
- The kernel of the linearized operator creates problems in solving the nonlinear problem
- This creates singularity in the moduli space, e.g. the football
- How to “desingularize”:

Theorem (Mazzeo–Z, ’17–’19)

The action of splitting cone points give the nonlinear model for the kernels of the linearized operator.

- Now we want to understand which metrics have a nontrivial kernel

Spectral theory of conical operators

- Laplacian of a complete manifold is L^2 self-adjoint
- On an incomplete manifold, we need to specify boundary conditions
- Von Neumann theory: there is a one-to-one correspondence of self-adjoint extensions of an operator with Lagrangian subspaces of boundary conditions
- e.g. d^2/dx^2 on $[0, 1]$, Dirichlet/Neumann/mixed boundary conditions give different extensions

A conical operator

- In our case, the operator is given by $\Delta_g - 2$ where g is a spherical conical metric
- Local coordinates:

$$g = dr^2 + \beta^2 \sin^2 r \, d\theta^2$$

$$\begin{aligned}\Delta_g &= -\partial_r^2 - \frac{\cos r}{\sin r}\partial_r - \frac{1}{\beta^2}\frac{1}{\sin^2 r}\partial_\theta^2 \\ &= -r^{-2}[(r\partial_r)^2 + \beta^{-2}\partial_\theta^2 + \mathcal{O}(r^2\partial_r, r\partial_\theta)]\end{aligned}$$

- Δ_g is a conical operator, i.e. $\Delta_g \in r^{-2}\mathrm{Diff}_b^2(\Sigma_{p_1,\ldots,p_n})$
- The eigenvalue 2 is a lower order term

Literature

- Mapping properties: [Cheeger, '79] [Brüning–Seeley, '85, '88] [Mazzeo, '91] [Seeley, '03] [Gil–Krainer–Mendoza, '06]
- Self-adjoint extensions of conical/wedge operators: [Gil–Krainer–Mendoza, '07, '13]
- Spectral geometry on flat conical surfaces: [Hillairet, '10] [Hillairet–Kokotov, 15] [Kokotov–Lagota, '19]
- Determinant of Laplacians: [Mooers, '99] [Loya–McDonald–Park, '05] [Gil–Loya, '08] [Sher, '15] [Kalvin–Kokotov, '17] [Nursultanov–Rowlett–Sher, '19]
- Scattering theory on conical manifold: [Melrose, Wunsch, Vasy, Baskin. . . , '00–]

Mapping properties and domains

- The minimal domain of Δ_g is the closure of $C_c^\infty(\Sigma \setminus \{p_1, \dots, p_n\})$ with respect to the graph norm
- The maximal domain is the L^2 dual of the minimal domain
- Self-adjoint extensions are mid-dimensional spaces between $\mathcal{D}^{\min}$ and $\mathcal{D}^{\max}$
- The simplest case: if $2\pi\beta \le 2\pi$, then locally

$$u \in \mathcal{D}^{\max} \iff \exists \tilde{u} \in \mathcal{D}^{\min}, a_0, b_0 \in \mathbb{C}, u = \tilde{u} + a_0 + b_0 \log r$$

where $\{\text{self adjoint extensions}\} \leftrightarrow \{\text{Lagrangians in } \mathbb{C}^2_{(a_0, b_0)}\} \simeq \mathbb{S}^2$

- The next simplest example: if $2\pi\beta \in (2\pi, 4\pi]$, then

$$u \in \mathcal{D}^{\max} \iff \exists \tilde{u} \in \mathcal{D}^{\min}, a_i, b_i \in \mathbb{C}, i = -1, 0, 1$$

$$u = \tilde{u} + a_0 + b_0 \log r + a_1 r^{\frac{1}{\beta}} e^{i\theta} + a_{-1} r^{\frac{1}{\beta}} e^{-i\theta} + b_1 r^{-\frac{1}{\beta}} e^{-i\theta} + b_{-1} r^{-\frac{1}{\beta}} e^{i\theta}$$

then self-adjoint extensions correspond to Lagrangians in $\mathbb{C}^6$

Two self-adjoint extensions

- We consider two different extensions
- Take the $2\pi\beta \in (2\pi, 4\pi]$ case for example
- The Friedrichs extension $\mathcal{D}^{\mathrm{Fr}}$: the only bounded extension

$$u = \tilde{u} + a_0 + a_1 r^{\frac{1}{\beta}} e^{i\theta} + a_{-1} r^{\frac{1}{\beta}} e^{-i\theta}$$

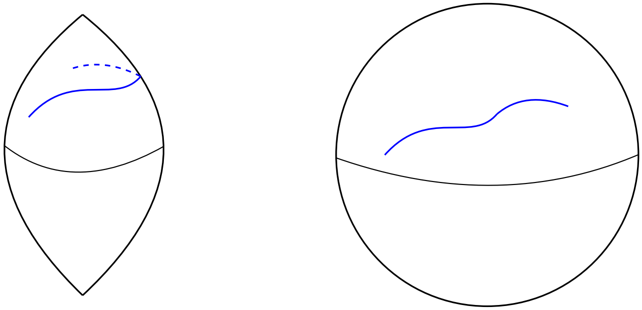
- Holomorphic extension $\mathcal{D}^{\mathrm{Hol}}$: take the holomorphic half of the coefficients

$$u = \tilde{u} + a_0 + a_1 r^{\frac{1}{\beta}} e^{i\theta} + b_1 r^{-\frac{1}{\beta}} e^{-i\theta}$$

- When all cone angles are less than 2π: two extensions are equal
- Translation surfaces: Dirichlet-To-Neumann isospectrality [Hillairet, '09]

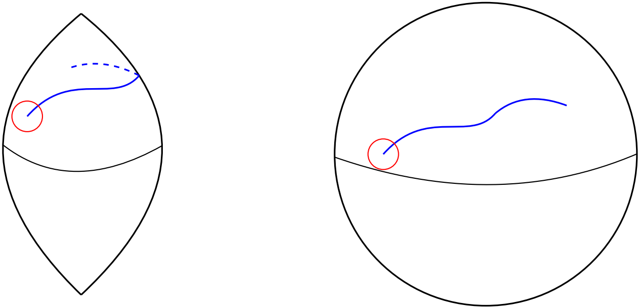
A special class of metrics

- We study the spectrum of a special class of metrics
- The classical way in complex analysis to understand conical metrics is through “developing maps”



A special class of metrics

- A special class of spherical metrics can be characterized in two ways
- The classical way: complex analysis through “developing maps”



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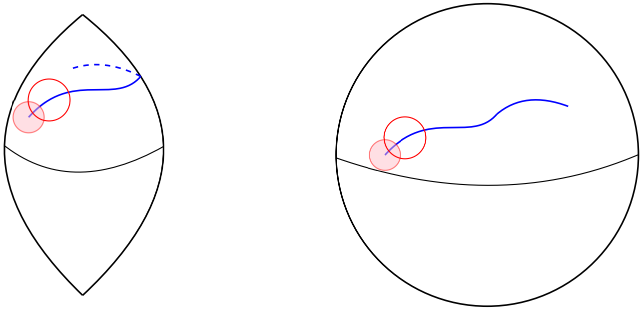

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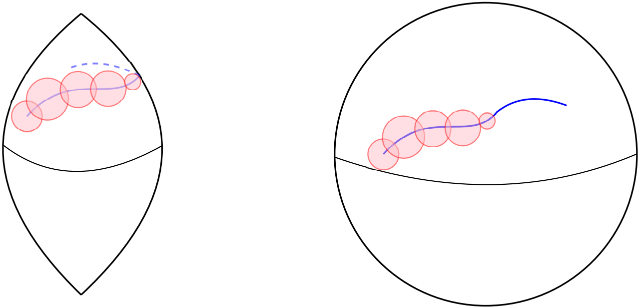

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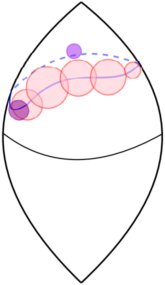
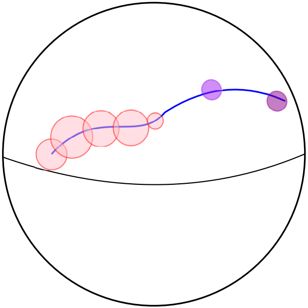




Formal definition

For a spherical conical metric g, there exists a (usually non-unique) multi-valued meromorphic map

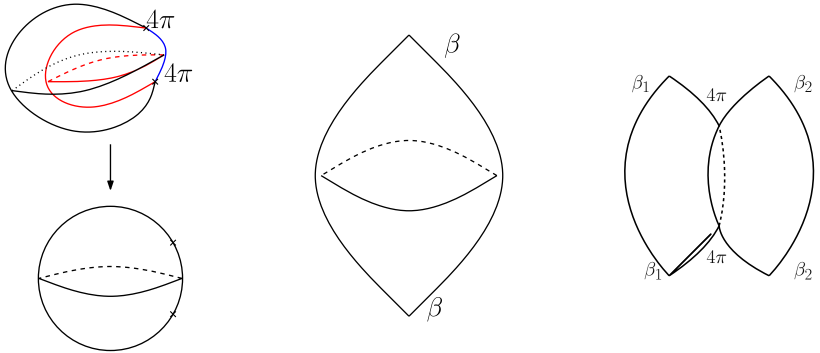
$$f : \Sigma \setminus \{p_1, \dots, p_n\} \to \mathbb{P}^1 = \overline{\mathbb{C}},$$

called a developing map of g with the three properties

1. (Pull-back) Denote by g_{st} the standard spherical metric, then $g = f^* g_{\mathrm{st}}$ on $\Sigma \setminus \{p_1, \dots, p_n\}$;
2. (Monodromy) The monodromy of f is contained in $\mathrm{PSU}(2) = \mathrm{SO}(3)$;
3. (Cone angle) Near angle $2\pi\beta_j$, the principal singular term of the Schwarzian derivative of f is given by $\frac{1-\beta_j^2}{2z^2}$.

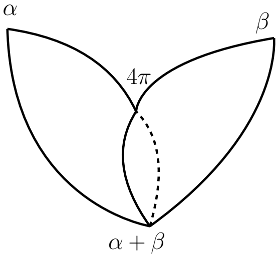
Monodromy

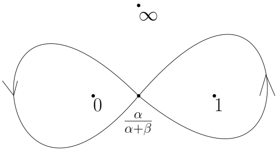
- The monodromy of developing maps of the same metric are in the same conjugacy class
- A metric is called reducible if there exists a developing map with monodromy in $\mathrm{U}(1)$
  [Umehara–Yamada, '03] [Chen-Wang–Wu–Xu, '15]
- Examples of reducible metrics:



What is special about reducible monodromy

- Each metric is associated with a meromorphic one-form
- Relation to the developing map: $\omega = df/f,\ f = C\exp\left(\int \omega\right)$
- Example: the “heart” is given by the form $(\frac{\alpha}{z} + \frac{\beta}{z-1})dz$
- This gives horizontal/vertical foliation, assembled by ribbon graphs





- Any reducible metric has a noncompact family of conformal dilations
- Such metrics have deformation rigidity [Z, ’19]

Relation to translation surfaces

- Each reducible metric on $\mathbb{S}^2$ corresponds to a half-infinite translation surface
- Take the meromorphic one-form ω, then $(\Sigma, |\omega|^2)$ gives a flat conical metric
- Equivalently: take the developing map $f : \Sigma^{**} \to \mathbb{C}$ and pull back the flat metric $|dz|^2/|z|^2$ on $\mathbb{C}$
- There are plenty of reducible metrics [Eremenko, '17]

Statement of the main theorem

Theorem

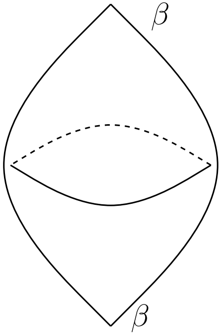
A spherical conical metric g has reducible monodromy if and only if $2 \in \text{spec}(\Delta^{\text{Hol}}) \cap \text{spec}(\Delta^{\text{Hol}})$.

- Equivalent phrase: reducible monodromy if and only if there is a real-valued eigenfunction $\phi \in \mathcal{D}^{\text{Hol}}$ with $\Delta_g \phi = 2\phi$
- One direction is easy: we generate such an eigenfunction from a good developing map f

$$\phi = \frac{1 - |f|^2}{1 + |f|^2}$$

- The other direction: not obvious!

An example: the spherical football



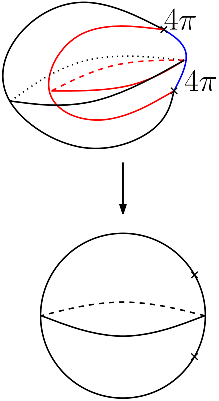
Spherical footballs

- Take coordinate z centered on the north pole
- The eigenfunction is given by

$$\phi = \frac{1 - |z|^{2\beta}}{1 + |z|^{2\beta}}$$

- Its gradient vector field is given by $-z\partial_z$
- This vector field corresponds to conformal dilations

Another example: branched covers



A double cover of the sphere, developing map $f = z^2$

- There are three 2-eigenfunctions (all from the pullback of the sphere), given by

$$\phi = \frac{1-|f|^2}{1+|f|^2}, \ \Re\frac{2f}{1+|f|^2}, \ \Im\frac{2f}{1+|f|^2}$$

- Corresponding gradient vector fields:

$$-\frac{1}{4}z\partial_z, \ (\frac{1}{8}z^{-1} - \frac{1}{8}z^3)\partial_z, \ i(\frac{1}{8}z^{-1} + \frac{1}{8}z^3)\partial_z.$$

Why number 2? (again)

- $\Delta - 2$ is the linearized operator of the Liouville equation
- The first nonzero eigenvalue of the standard sphere
- eigenvalue isoperimetric problem: Li–Yau upper bound of the first (normalized) eigenvalue of any smooth metric on a genus 0 surface, where 2 is only achieved by the sphere
- If a smooth manifold is Kähler:

$$\mathrm{Ric} \geq \mu > 0 \Rightarrow \lambda_1 \geq 2\mu. \text{ (Lichnerowicz type estimate)}$$

- If $n = 2$, equality only achieved by the sphere

Bochner's technique in the small-angle case

- The Lichnerowicz type argument still works for a spherical conical metrics if all the cone angles are less than 2π
- In this case one still gets $\lambda_1 \geq 2$ [Luo–Tian, '92] [Mazzeo–Weiss, '15]
- $\lambda_1 = 2$ if and only if g is a football
- Proof idea: using Bochner's identity for the complex gradient vector field X

$$\nabla^* \nabla^{(0,1)} X = \frac{1}{2}(\lambda - 2)X + (X - \mathrm{Ric} X)$$

- Then apply integration by parts to get a holomorphic vector field with enough vanishing points
- For large cone angles, this argument would not work any more for $u \in \mathcal{D}^{\mathrm{Fr}}$

Our proof: Lichnerowicz technique revisited

- Bochner's identity still holds

$$\nabla^*\nabla^{(0,1)}X = \frac{1}{2}(\lambda - 2)X + (X - \mathrm{Ric}X) = 0$$

- The question is whether integration by parts still works
- In general, the answer is no. (In particular, if $u \in \mathcal{D}^{\mathrm{Fr}}$)
- However, when $u \in \mathcal{D}^{\mathrm{Hol}}$, the decay is just enough
- This gives a meromorphic vector field X

From a meromorphic vector field to developing maps

- Once we obtain a meromorphic vector field, it generates a (correctly rescaled) dual meromorphic 1-form ω and a map

$$f = \exp\left(\int \omega\right) : \Sigma^* \to \mathbb{C}^*$$

- Then show that f is one of the developing maps which extends to the punctured surface
- We also have a dimension counting statement to pick up the trivial monodromy

Theorem (Xu–Z, '19)

The dimension of such eigenfunctions is either 1 or 3. In particular, the dimension is 3 if and only if it is a branched cover.

Thank you for your attention!