Nodal degeneration of hyperbolic metrics and application to the Weil-Petersson metric on $\mathcal{M}_{g,n}$

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Joint work with Richard Melrose

Moduli space $\mathcal{M}_{g,n}$ and universal curve $\mathcal{C}_{g,n}$

- Moduli space $\mathcal{M}_g$ of genus g Riemann surface, $g \geq 2$
- Complex structure $\leftrightarrow$ metric structure
- $\mathcal{M}_{g,n}$ moduli space of punctured Riemann surfaces with genus g and n ordered distinct marked points
- Stable curve: $2g - 2 + n > 0$
- Universal curve $\mathcal{C}_{g,n}$ fibers over $\mathcal{M}_{g,n}$
- $\mathcal{C}_{g,n}$ is identified with $\mathcal{M}_{g,n+1}$
- Each fiber of $\mathcal{C}_{g,n}$ carries a finite area hyperbolic metric
- The hyperbolic metric varies smoothly over the fibers
- Weil-Petersson metric defined using the hyperbolic metric

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Deligne–Mumford compactification of $\mathcal{M}_{g,n}$

- The space $\mathcal{M}_{g,n}$ is not compact.
- Deligne–Mumford compactification $\overline{\mathcal{M}_{g,n}}$ corresponds to adding nodal crossing divisors
- Singular fibration $\overline{\mathcal{M}_{g,n+1}} \to \overline{\mathcal{M}_{g,n}}$

Questions

- How does the Weil–Petersson metric behave near the divisors on $\overline{\mathcal{M}_{g,n}}$?
- How does the canonical hyperbolic metric behave under nodal degeneration?

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Degeneration I: pinching geodesics

In the compactification, nodal curves are added corresponding to pinching geodesics.

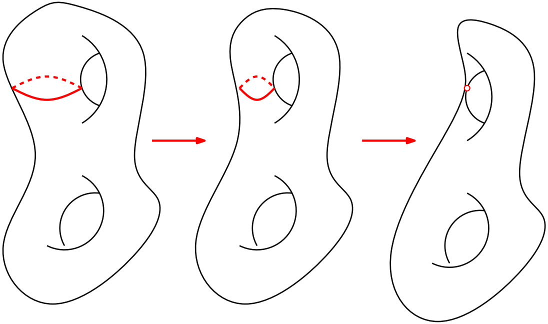

Figure: Degenerating surfaces with a geodesic cycle shrinking to a point

Nodal crossing divisors

The previous picture might be misleading: the singular surface has a transversal crossing

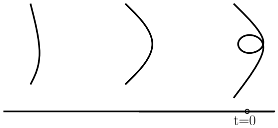

Figure: Transversal crossing of universal curve

Locally the behavior is given by the plumbing variety

Definition

A plumbing variety is given by the following singular fibration

$$\psi : \mathbb{C}^2 \ni (z, w) \longrightarrow t = zw \in \mathbb{C}.$$

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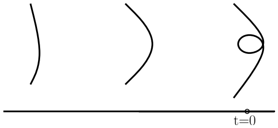

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Divisors in $\overline{\mathcal{M}_g}$

- The "boundary" $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is a union of normally intersecting, self-intersecting divisors
- Pinching one geodesic gives a pair of nodal points
- If the fiber has k pairs of nodal points, it lies on the intersection of k local divisors, i.e. locally a k-fold intersection of $\mathcal{M}_{g-1,2}$
- The arithmetic genus $\mathcal{G} = 2g + n$ stays the same

Degeneration II: pointed moduli space $\mathcal{M}_{g,n}$

- Another degeneracy: marked points may collide
- Example of $\mathcal{M}_{0,4}$ of $\mathbb{P}^1$ with 4 points: $\{0, 1, \infty, t\}$ vs $\{0, 1/t, \infty, 1\}$
- Deligne-Mumford compactification separates the "colliding" points by adding nodal spheres
- A divisor in $\overline{\mathcal{M}}_{g,n}$ is represented by sequence of marked surfaces (with possible loops) connected by nodal crossings
- Singular fibration of $\overline{\mathcal{M}}_{g,n+1}$ over $\overline{\mathcal{M}}_{g,n}$ by dropping the last point and possibly pinching unstable components

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Nodal curves

Picture source: http://www.partyballoonanimals.co.uk/wp-content/themes/alexandria-child/images/balloon-animal.png

Multi-Lefschetz fibration

To characterize the fibration of universal curve over moduli space, we define multi-Lefschetz fibrations

- $\phi : M^{n+1} \to Z^n$ is a regular fibration except on finitely many points in M
- Near each of the nodal points there are holomorphic coordinates in which the map is the product of a Lefschetz map and a projection
- Local chart is given by

$$\phi : (z, w, \eta_1, \ldots \eta_{n-1}) \mapsto (t = zw, \eta_1, \ldots \eta_{n-1})$$

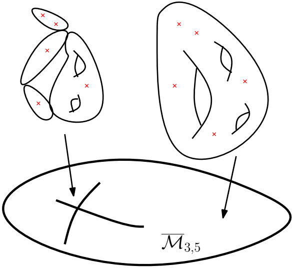

Figure: Universal curve fibers over the compactified moduli space $\overline{\mathcal{M}}_{3,5}$

Cotangent bundle of $\mathcal{M}_{g,n}$

The cotangent bundle of $\mathcal{M}_{g,n}$ is naturally identified with the bundle of holomorphic quadratic differentials on the fibers of $\mathcal{M}_{g,n+1}$

$$q : T^{1,0}\mathcal{M}_{g,n} \simeq Q\mathcal{M}_{g,n}.$$

Using this identification, the Weil-Petersson (co-)metric is defined by

$$G_{WP}(\zeta_1, \zeta_2) = \int_{fib} \frac{\zeta_1 \overline{\zeta_2}}{\mu_H},\ \zeta_1,\ \zeta_2 \in Q_p,\ p \in \mathcal{M}_{g,n}$$

where μ_H is the area form of the fiber hyperbolic metric and the integrand itself may be identified as a fiber area form.

Literature review

- Hyperbolic metrics on nodal crossing: [Wolpert, 1990] [Wolf, 1991] [Obitsu–Wolpert, 2009]
- Geometry of moduli space: [Bers, 1973, 1974] [Deligne–Mumford, 1979] [Robbin–Salamon, 2006]
- Weil–Petersson metric asymptotics: [Masur, 1976] [Wolpert, 2001, 2015] [Mazzeo–Swoboda, 2015]
- Problems related to Weil–Petersson metric: [Wolpert, 1982, 1986, 1990, 1992, 2008, 2012] [Takhatajan–Zograf, 1991] [Yamada, 2004] [Liu–Sun–Yau, 2004, 2005] [Obitsu–To–Weng, 2008] [Ji–Mazzeo–Müller–Vasy, 2014] [Gell-Redman–Swoboda, ongoing]

Previous results

Regarding the degenration of hyperbolic metrics, Obitsu and Wolpert gave an expansion of the canonical metric up to 3rd order:

Theorem[Obitsu–Wolpert, 2009]

Let ds^2_{cc} be the hyperbolic metric on the degenerated family R_t with m vanishing cycles, Δ the associated Laplacian, and ds_{pl} the plumbing metric that comes from gluing $ds^2_{P_t}$ with the regular part, then the metric has the following expansion

$$ds^2_{cc} = ds^2_{pl}\Big(1 - \frac{\pi^2}{3}\sum_{j=1}^{m}(\log|t_j|)^{-2}(\Delta+2)^{-1}(\Lambda(z_j)+\Lambda(w_j)) + O\big(\sum(\log|t_j|)^{-4}\big)\Big)$$

where the function Λ is given by $\Lambda(z_j) = (s_z^4 \chi_{\psi^{-1}\mathbb{D}_{1/2}})_{s_z}, \quad s_z = \log|z_j|$.

Resolution of canonical fiber metrics

We improved the result of Obitsu–Wolpert, gave the complete expansion and showed that under a suitable resolution:

$$\begin{array}{ccc} \widehat{M} & \longrightarrow & M \\ \downarrow \hat{\psi} & & \downarrow \psi \\ \hat{Z} & \longrightarrow & Z \end{array}$$

- The real fibration map is a b-fibration
- The fiber metric is conformal to a smooth metric on ${}^{L}T\widehat{M}$ a rescaling of the fiber tangent bundle
- The conformal factor is log-smooth

Plumbing metric

Our proof starts with the local model.

- Consider the plumbing variety

$$P = \{(z, w, t) \in \mathbb{C}^3; zw = t,\ |z| \leq 1,\ |w| \leq 1,\ |t| \leq 1/2\} \longrightarrow \mathbb{D}_{\frac{1}{2}} = \{t \in \mathbb{C}; |t| \leq 1/2\}.$$

Plumbing metric on each fiber

$$g_{pl}^{(t)} = \left(\frac{\pi \log |z|}{\log |t|} \csc \frac{\pi \log |z|}{\log |t|} \right)^2 g_0,$$

$$g_0 = \left(\frac{|dz|}{|z| \log |z|} \right)^2$$

- $g_{pl}^{(t)} \to g_0$ as $t \to 0$
- Symmetric with the change of $w = t/z$
- Fiber curvature $= -1$

Step 1: resolving the angular variable

To make g_{pl} smooth at $t = 0$, we need to first blow up $\{z = 0\}$ and $\{w = 0\}$ which are transversal:

$$P_{\bar{\partial}} := [P; \{z = 0\} \cup \{w = 0\}].$$

$$P_{\bar{\partial}} = \{(|z|, |w|); 0 \leq |z|, |w| \leq 1,\ |z||w| \leq \frac{1}{2}\} \times \mathbb{S}_z \times \mathbb{S}_w.$$

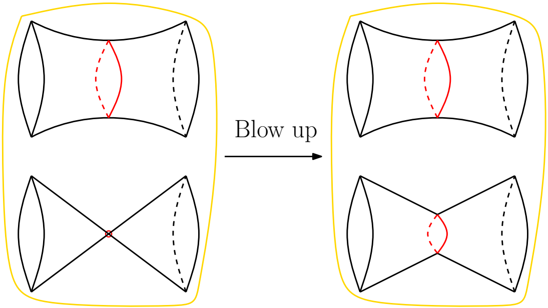

Figure: Blow up of $\{z = 0\} \cup \{w = 0\}$

Step 2: Logarithmic blow up

We do a "logarithmic blow up" to the space obtained above:

$$[P; \{z = 0\}_{\log} \cup \{w = 0\}_{\log}].$$

This step introduces smooth functions $1/\log|z|$ and $1/\log|w|$:

$$\operatorname{ilog} z = \frac{1}{\log \frac{1}{|z|}}, \ \operatorname{ilog} w = \frac{1}{\log \frac{1}{|w|}}$$

Step 3

After change of variable, the metric becomes

$$g_{pl}^{(t)} = \frac{\pi^2(\text{ilog}\, t)^2}{\sin^2\left(\frac{\pi\, \text{ilog}\, t}{\text{ilog}\, w}\right)} \left(\frac{(d\, \text{ilog}\, w)^2}{(\text{ilog}\, w)^4} + d\theta_w^2 \right)$$

where

$$\text{ilog}\, t = \frac{\text{ilog}\, z\, \text{ilog}\, w}{\text{ilog}\, z + \text{ilog}\, w} = \frac{\text{ilog}\, w}{1 + \frac{\text{ilog}\, w}{\text{ilog}\, z}}$$

is not a smooth function.

We blow up, radially, the corner formed by the intersection of the two logarithmic boundary faces

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Resolved space $\widehat{M}$

We consider the following glued space of $\widehat{M} = (M \setminus P) \cup P_{\mathrm{mr}}$:

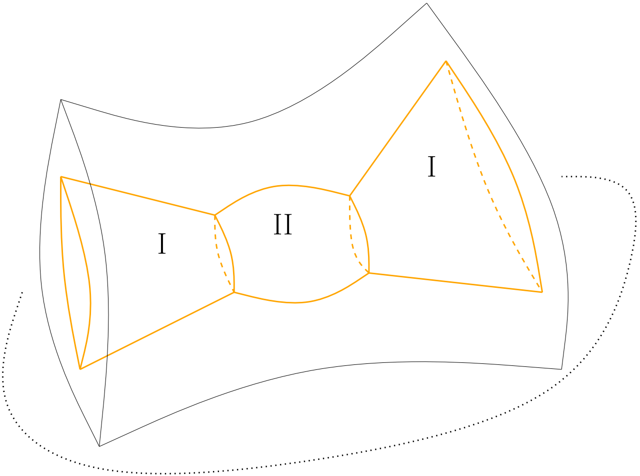

Figure: Final resolved space $\widehat{M}$

Result on Lefschetz fibration

Now we have a b-fibration:

$$\begin{array}{ccc} \widehat{M} & \longrightarrow & M \\ {\scriptstyle \hat{\psi}}\downarrow & & \downarrow{\scriptstyle \psi} \\ \hat{Z} & \longrightarrow & Z \end{array}$$

- The fiber tangent bundle on $\widehat{M}$ after log rescaling is the bundle which the metric lives.
- Take a smooth hermitian metric on $T\widehat{M}$, we solve for the conformal factor.

Theorem[Melrose–Z, 2015]

The fiber metrics of fixed constant curvature on a Lefschetz fibration extend to a continuous Hermitian metric on ${}^{L}T\widehat{M}$ which is related to a smooth Hermitian metric on this complex line bundle by a log-smooth conformal factor.

Curvature equation on $\widehat{M}$

Curvature equation for conformal factor: if $g = e^{2f} g_0$, then

$$R(g)e^{2f} = \Delta_{g_0} f + R(g_0),$$

which in our case is

$$\Delta_{g_{pl}} f + R(g_{pl}) = -e^{2f}.$$

The linearization of this equation:

$$\Delta_{g_{pl}} f + R(g_{pl}) = -1 - 2f.$$

Solvability of $\Delta + 2$ on $\widehat{M}$

We solve the linearized equation

$$(\Delta + 2)u = f \in O\left((\mathrm{ilog}\, t)^2\right)$$

on the space $\widehat{M}$.

- Two boundary faces: face I is the regular Riemann surface and face II is the one introduced in the last step
- Indicial roots: $\{1, -2\}$ for face I, and $\{-1, 2\}$ for face II
- Invertibility of $\Delta + 2$ on suitable weighted Sobolev spaces
- Appearance of extra log terms

Log-smoothness of a genuine solution

Solve iteratively to get a formal expansion for the curvature equation

$$\Delta_{g_{pl}} f + R(g_{pl}) = -e^{2f}$$

where f has the following expansion

$$f \sim \sum_{k \geq 2}^{\infty} g_k$$

- g_k has a factor of $(\text{ilog}\, t)^k$;
- Generally with logarithmic factors.

Then we use a perturbation argument to show the existence of a genuine solution.

Multiple shrinking curves

Now we generalize the Lefschetz fibration to multi-Lefschetz fibration.

- Cusp metric locally near the nodes
- Blow up at every node to get front face $\mathrm{II}_1, ..., \mathrm{II}_n$

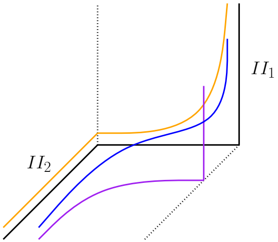

Figure: Universal curves undergoing degeneration of two geodesics

Iteration for solving the curvature equation

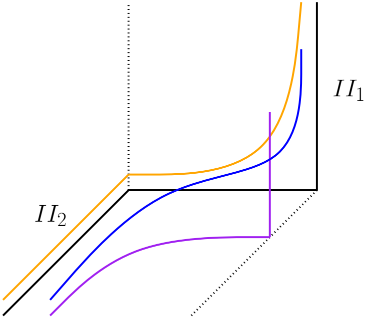

Figure: Universal curves undergoing degeneration of two geodesics

- Start with curves with cusps
- Solve the linear equation $(\Delta + 2)f = O\left((\text{ilog}\, t_1 \,\text{ilog}\, t_2 \ldots \text{ilog}\, t_n)^2\right)$
- Log terms appear in linear growth

Quadratic holomorphic differential: log-cotangent bundle

- The cotangent space of $\mathcal{M}_g$ at a regular point consists of holomorphic quadratic differentials
- On the divisor, it contains meromorphic ones with poles at most degree two
- Identified with smooth sections of the log cotangent bundle

This gives us a way to find the complete expansion of Weil–Petersson metric.

Log geometry

- There is a well-defined 'logarithmic' complex tangent ${}^{L}T^{(1,0)}M$ and cotangent bundle ${}^{L}\Lambda^{(1,0)}M$
- The spaces of locally holomorphic sections of ${}^{L}T^{(1,0)}M$ are the holomorphic vector fields which are tangent to all the local divisors.
- In admissible local coordinates ${}^{L}T^{(1,0)}M$ is locally spanned by the holomorphic vector fields $t\partial_t$ and ∂_{z_k}.
- The complex dual of this bundle, ${}^{L}\Lambda^{(1,0)}M$, is locally spanned in these coordinates by the dt/t and dz_k.

Definition

Logarithmic cotangent bundle ${}^{L}\Lambda^{(1,0)}\overline{\mathcal{M}}_{g,n}$ is defined to be the sheaf of differentials which are logarithmic across the exceptional divisors.

Log cotangent bundle

- We show that ${}^L\Lambda^{(1,0)}\overline{\mathcal{M}}_{g,n}$ is naturally isomorphic to an appropriate bundle of holomorphic quadratic differentials on the fibres (including the singular ones above the divisors) of $\overline{\mathcal{M}}_{g,n+1}$. This extends the proof of Robbin and Salamon.
- Dimension counting: the dimension of moduli space $\mathcal{M}_{g,n}$ is $3g-3+n$.
- Dimension of cotangent space approaching one component of divisor $\mathcal{M}_{g-1,n+2}$: $3(g-1)-3+(n+2)+1=3g-3+n$, where the extra 1 comes from the residual on the nodal points

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Main theorem about WP metric on $\overline{\mathcal{M}_g}$

A resolution of the complex compactification is given by

$$\begin{array}{ccc} \widehat{\mathcal{M}}_{g,n+1} & \xrightarrow{\beta} & \overline{\mathcal{M}}_{g,n+1} \\ \downarrow \hat{\phi} & & \downarrow \overline{\phi} \\ \hat{\mathcal{M}}_{g,n} & \xrightarrow[\beta]{} & \overline{\mathcal{M}}_{g,n} \end{array}$$

Lifting the log cotangent bundle to this resolution, we use the push forward theorem to show that the Weil-Petersson metric is log-smooth on ${}^{L}\Lambda^{(1,0)}\widehat{\mathcal{M}}_{g,n+1}$.

$$G_{WP}(\zeta_1, \zeta_2) = \int_{fib} \frac{\zeta_1 \overline{\zeta_2}}{\mu_H}, \ \zeta_1, \ \zeta_2 \in Q_p({}^{L}\Lambda^{1,0}\widehat{\mathcal{M}}_{g,n+1}), \ p \in \hat{\mathcal{M}}_{g,n}$$

Application I: Expansion of shortest geodesics

Corollary

The length of the shortest geodesic under degeneration is a polyhomogeneous function of ilog $|t|$.

- In the plumbing model, the shortest geodesic is given by the circle $|z_0| = \sqrt{|t|}$
- $l_{pl}(t) = 2\pi^2 / \log|t|$
- Rotational symmetry of the actual hyperbolic metric (up to infinite order)
- The minimizing curve still occurs in the circle $|z_0| = \sqrt{|t|}$
- $l_{hp}(t) = e^{f(|z_0|,t)} l_{pl}(t) + O(t^\infty)$

Application II: Takhatajan–Zograf metric

- For a punctured Riemann surface, the first Chern class on the degree k line bundle is related to the WP metric by

$$c_1(\bar{\partial}_k) = \frac{6k^2 - 6k + 1}{12\pi^2}\omega(g_{WP}) - \frac{1}{9}\omega(g_{TZ})$$

- Takhatajan–Zograf metric is given by

$$(q_1, q_2)_{TZ} = \int_{fib} \sum_i \frac{E_i^{-1} q_1 \bar{q}_2}{\mu_H}$$

  where E_i is the Eisenstein series at the i-th puncture

- We obtain the expansion of TZ metric and its degenerating behavior.

Thank you for your attention!